ON A FAMILY OF CLASSES OF FUNCTIONS OF THE ALGEBRA OF LOGIC AND THEIR REALIZATION IN THE CLASS OF Π-CIRCUITS

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Abstract

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MATHEMATICS

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ON A FAMILY OF CLASSES OF FUNCTIONS OF THE ALGEBRA OF LOGIC AND THEIR REALIZATION IN THE CLASS OF Π-CIRCUITS

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It is known (^{1,2}) that most functions of the algebra of logic have a complex circuit realization both in the class of parallel-series circuits (Π-circuits) and in the class of all contact circuits. Therefore the question arises of finding classes of functions that are realized more simply. This question has been investigated by many authors ($^{2-5}$).

Here we consider a family of certain such classes, namely the classes $R_{n,k}$ of functions depending on n arguments and taking the value 1 on exactly k sets of values of the n arguments, and we investigate the possibility of realizing functions from these classes by Π-circuits. This work is a continuation of the work of G. N. Igosheva, carried out in 1954/55, who proposed a certain method of circuit synthesis for functions from $R_{n,k}$ and obtained upper estimates for the minimal number of contacts sufficient for realizing any function from $R_{n,k}$ in the case $k = 2, 3, 4, 5$. (These estimates are, respectively: $2n$, $[\frac{7}{3}n]$, $[\frac{20}{7}n]$, $[\frac{53}{15}n]$.) In the present paper another algorithm is described for realizing functions from these classes, and it is shown that even with a certain growth of k the number of contacts in Π-circuits realizing functions from $R_{n,k}$ does not asymptotically exceed $2n$.

As is known, realization of a function of the algebra of logic by a parallel-series circuit is equivalent to writing it by means of a formula in which the signs of the arguments x_i and the signs &, $\vee$, $^{-}$ are used, where the signs $^{-}$ occur only over the signs of arguments. In what follows we shall consider only formulas of this kind.

Let $L(f)$ denote the minimal number of variable signs sufficient for writing the function f by a formula of the kind indicated above. Introduce the function

$$L_k(n) = \max L(f)$$

(the maximum is taken over all functions from $R_{n,k}$)*.

Theorem. $L_k(n) \leq 2n + k2^{k-1}$.

Proof. Let an arbitrary function f from $R_{n,k}$ be given by a table. We shall consider only those sets (of values of the arguments) on which the function is equal to 1 (k of them). The matrix whose rows are these sets will henceforth be called the **matrix of the given function**. By replacing some x_i by $\overline{x_i}$ one may arrange that the first row of the matrix consist only of ones, and by permuting the columns and renumbering the arguments—that equal columns stand next to one another. Let the matrix thus obtained correspond to a function f'. We divide the columns of the matrix into groups that are identical among themselves (the number of groups does not exceed 2^{k-1}).

Let now the arguments $x_1, x_2, \dots, x_{n_1}$ correspond to the 1st group of columns, …, $x_{n_{l-1}+1}, \dots, x_{n_l}$ correspond to the l-th (last) group of columns.

Consider the functions

$$f_1 = \&_{i=1}^{l} \left[\left(\&_{m=n_{i-1}+1}^{n_i} x_m \right) \vee \left(\&_{m=n_{i-1}+1}^{n_i} \overline{x}_m \right) \right], \qquad n_0 = 0,$$

and

$$f_2 = \bigvee_{p=1}^{k} \&_{i=1}^{l} x_{n_i}^{\sigma_{p,n_i}},$$

where σ_{p,n_i} is the element of the transformed matrix standing in the p-th row in the n_i-th column; x^0 denotes $\overline{x}$, x^1 denotes x. We shall show that $f_1 \& f_2 = f'$.

Indeed: consider the matrix of the function $\varphi = f_1 \& f_2$. It is easy to see that it will contain those and only those rows of the matrix f_1 which are simultaneously rows of the matrix f'.

Obviously, $L(f_1) \leqslant 2n$ and $L(f_2) \leqslant k2^{k-1}$, and, consequently, $L(f) = L(f') \leqslant 2n + k2^{k-1}$. Thus the theorem is proved.

Corollaries.

1. Let a sequence be given

$$k_2, \dots, k_n, \dots \tag{*}$$

If for some function $\varphi(n)$, $\varphi(n) \to \infty$,

$$k_n \leqslant \lg_2 n - \lg_2 \lg_2 n - \varphi(n),$$

then

$$L_{k_n}(n) \sim 2n.$$

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* In other words, $L_k(n)$ is the minimal number of contacts sufficient for realizing any function from $R_{n,k}$ in the class of Π-circuits.

The upper bound follows directly from the theorem, and the lower bound from the fact that, for the indicated values, there exist functions from $R_{n,k}$ that are not monotone in any of their arguments*.

2. If for the sequence $(*)$

$$\varlimsup_{n\to\infty} \frac{k_n}{\lg_2 n} < N$$

(N is a natural number), then

$$L_{k_n}(n) < 2Nn.$$

3. If

$$\frac{k_n}{\lg_2 n} \to \infty,$$

then

$$L_{k_n}(n) \leqslant \frac{2k_n n}{\lg_2 n}(1 + o(1)).$$

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* The function $f_1(x_1, \dots, x_n)$ is called nonmonotone in the argument x_i if it satisfies neither of the inequalities

$$f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \leqslant f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n),$$

$$f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \geqslant f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n).$$

Note: Figure translations are in progress. See original paper for figures.