THE CURVATURE TENSOR AND SOME TYPES OF SPACES OF A SYMMETRIC ALMOST SYMPLECTIC CONNECTION

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1957

SovietRxiv

Abstract

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MATHEMATICS

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THE CURVATURE TENSOR AND SOME TYPES OF SPACES OF A SYMMETRIC ALMOST SYMPLECTIC CONNECTION

(Presented by Academician I. G. Petrovskii on 11 VI 1957)

1. The curvature tensor $B_{ij,kl} = a_{l\alpha} R_{ij,k}^{\cdot\cdot\alpha}$ of a space of symmetric almost symplectic connection*, along with the properties

$$B_{ij,kl} + B_{ji,kl} = 0, \tag{1}$$

$$B_{ij,kl} + B_{jk,il} + B_{ki,jl} = 0, \tag{2}$$

which follow directly from the general properties of the tensor $R_{ij,k}^{\cdot\cdot\alpha}$, possesses a number of special properties, among which one should first of all mention the property

$$B_{lj,ik} + B_{jl,ki} + B_{ki,jl} + B_{ik,lj} = 0. \tag{3}$$

Indeed, from the identity

$$\frac{\partial T_{ijk}}{\partial x^l} - \frac{\partial T_{jkl}}{\partial x^i} + \frac{\partial T_{kli}}{\partial x^j} - \frac{\partial T_{lij}}{\partial x^k} = 0, \tag{4}$$

which is verified directly by substituting

$$T_{ijk} = \frac{1}{3}\left(\frac{\partial a_{ij}}{\partial x^k} + \frac{\partial a_{jk}}{\partial x^i} + \frac{\partial a_{ki}}{\partial x^j}\right), \tag{5}$$

it follows that

$$\nabla_l T_{ijk} - \nabla_i T_{jkl} + \nabla_j T_{kli} - \nabla_k T_{lij} = 0. \tag{6}$$

Hence, in view of the fact that

$$\nabla_k a_{ij} = T_{ijk}, \tag{7}$$

we obtain

$$\nabla_l \nabla_j a_{ki} - \nabla_j \nabla_l a_{ki} = \nabla_i \nabla_k a_{lj} - \nabla_k \nabla_i a_{lj}. \tag{8}$$

But, on the other hand,

$$\nabla_{[l} \nabla_{j]} a_{ki} = R_{lj,k}^{\cdot\cdot\alpha} a_{\alpha i} + R_{lj,i}^{\cdot\cdot\alpha} a_{k\alpha} = B_{lj,ik} - B_{lj,ki}. \tag{9}$$

Relation (9), in view of the validity of (8), proves identity (3). If we denote

$$\Pi_{lj,ki} = \frac{1}{2} \left(B_{lj,ik} - B_{lj,ki} \right), \tag{10}$$

then, by virtue of (9) and (7), we have

$$\Pi_{lj,ki} = \frac{1}{2} \left(\nabla_l T_{jki} - \nabla_j T_{lki} \right). \tag{11}$$

* For the definition of the space under consideration, see (1).

or

$$\begin{aligned} \nabla_l T_{jki} = 2\Pi_{lj,ki} + \nabla_j T_{lki} = 2\Pi_{lj,ki} + 2\Pi_{jk,il} + \nabla_k T_{jil} = \\ = 2\Pi_{lj,ki} + 2\Pi_{jk,il} + 2\Pi_{kl,ji} + \nabla_l T_{kji}. \end{aligned} \tag{12}$$

Hence

$$\nabla_l T_{jki} = \Pi_{l(j,ki)}. \tag{13}$$

The last relation, by virtue of (10), (1), and (2), takes the form

$$\nabla_l T_{jki} = \frac{1}{2} \left(B_{jk,li} + B_{ki,lj} + B_{ij,lk} \right). \tag{14}$$

2. Let us now consider those spaces of symmetric almost symplectic connection for which the elements of the Lie algebra of the holonomy group associated with elementary cycles belong to the Lie algebra of the symplectic group, i.e., for which

$$\nabla_l \nabla_j a_{ki} - \nabla_j \nabla_l a_{ki} = 0. \tag{15}$$

Hence, by virtue of (7), (11), and (13), we have

$$\nabla_l \nabla_j a_{ki} = 0. \tag{16}$$

Obviously, conversely as well, (16) implies (15). We note that from (10) and (11) it follows that spaces of the indicated type are characterized by the symmetry of the curvature tensor in the last two indices

$$B_{ij,kl} = B_{ij,lk}, \tag{17}$$

and consequently (14) gives

$$B_{ij,kl} + B_{jl,ki} + B_{li,kj} = 0. \tag{18}$$

Condition (16) is satisfied by spaces of symmetric symplectic connection ($T_{ijk} = 0$), and also by locally flat almost symplectic spaces ($B_{ij,kl} = 0$).

Let us give an example of a space satisfying relation (16), but different from the spaces indicated above. Put

$$a_{ij} = \alpha_{ijk} x^k + \beta_{ij}, \tag{19}$$

where $\alpha_{ijk} = \text{const}$, $\beta_{ij} = \text{const}$, $\alpha_{ijk} = -\alpha_{jik} = \alpha_{jki}$, $\beta_{ij} = -\beta_{ji}$. Suppose, in addition, that the following conditions hold: 1) $\alpha_{ijk} \neq 0$ if and only if at least one of the indices i, j, k is equal to one; 2) $\gamma_{111} \neq 0$, the remaining $\gamma_{ijk} = 0$.

We have

$$\Gamma^i_{jk} = a^{il} \left(\frac{\partial a_{lk}}{\partial x^j} - \frac{\partial a_{jl}}{\partial x^k} + \gamma_{ljk} \right); \tag{20}$$

hence, by virtue of (19) and condition 2), it is clear that nonzero Γ^i_{jk} can occur only among

$$\Gamma^i_{11} = a^{i1} \gamma_{111} \quad (i \geq 2). \tag{21}$$

Consequently, nonzero $\nabla_l T_{ijk}$ can occur only among

$$\nabla_1 T_{ljk} = -\alpha_{mjk} \Gamma^m_{11} \quad (1 \neq j \neq k \neq 1). \tag{22}$$

But they too, by virtue of condition 1), are equal to zero. Thus, $\nabla_l T_{ijk} = 0$, but $T_{ijk} = \alpha_{ijk} \neq 0$.

Thus, the example constructed is not a space of a symplectic connection, but, naturally, $|a_{ij}| \neq 0$, since the choice of β_{ij} is at our disposal. Further, from the formula

$$B_{ij,kl} = \frac{1}{3}\left(\frac{\partial^2 a_{jl}}{\partial x^k \partial x^i} - \frac{\partial^2 a_{il}}{\partial x^k \partial x^j}\right) + \frac{1}{3}\left(\frac{\partial \gamma_{lik}}{\partial x^j} - \frac{\partial \gamma_{ljk}}{\partial x^i}\right) +$$

$$+ a_{\alpha\beta}\left(\Gamma^{\alpha}_{ki}\Gamma^{\beta}_{lj} - \Gamma^{\alpha}_{kj}\Gamma^{\beta}_{li}\right) - T_{l\alpha j}\Gamma^{\alpha}_{ik} + T_{l\alpha i}\Gamma^{\alpha}_{jk}, \tag{23}$$

by virtue of (19) and conditions 1) and 2), it follows that

$$B_{12,11} = \frac{1}{3}\frac{\partial \gamma_{111}}{\partial x^2} + \alpha_{21\beta} a^{\beta 1}\gamma_{111}, \tag{24}$$

i.e. $B_{12,11} \neq 0$ and, consequently, the example constructed is not a flat almost symplectic space.

3. Let us now consider spaces of symmetric almost symplectic connection characterized by the property

$$b_i = \frac{1}{2} a^{jk} T_{jki} = 0. \tag{25}$$

These are spaces with covariantly constant volume $\sqrt{\det \|a_{ij}\| \det \|X^k_{(l)}\|}$.

Let us note that, for $T_{ijk} \neq 0$, condition (25) will be stronger than the condition of equiaffinity. Indeed, the condition of equiaffinity

$$R_{ij,\alpha}{}^{\alpha} = \frac{\partial \Gamma^{\alpha}_{i\alpha}}{\partial x^j} - \frac{\partial \Gamma^{\alpha}_{j\alpha}}{\partial x^i} = 0, \tag{26}$$

by virtue of (20), takes the form

$$\frac{\partial\left(a^{\alpha\beta}\dfrac{\partial a_{\beta i}}{\partial x^{\alpha}}\right)}{\partial x^j} = \frac{\partial\left(a^{\alpha\beta}\dfrac{\partial a_{\beta j}}{\partial x^{\alpha}}\right)}{\partial x^i} = 0, \tag{27}$$

but from $b_i = \frac{1}{2} a^{pq} T_{pqi}$ we obtain

$$a^{\alpha\beta}\frac{\partial a_{\beta i}}{\partial x^{\alpha}} = 3b_i + \frac{\partial \ln\sqrt{a}}{\partial x^i}, \tag{28}$$

where $a = \det \|a_{ij}\|$, and, consequently, (27) gives

$$\frac{\partial b_i}{\partial x^j} = \frac{\partial b_j}{\partial x^i}, \tag{29}$$

i.e. $b_i = \operatorname{grad}\varphi$. Obviously, conversely, if the vector b_i is a gradient, then the space is equiaffine. Finally, we note that the condition $b_i = 0$ entails (29) and, consequently, equiaffinity, but not conversely. Examples of equiaffine spaces that are not spaces with invariant volume are flat almost symplectic spaces, as well as curved spaces obtained from flat ones by the choice of the object γ_{ijk}.

Let us show that, in the case of 2 and 4 dimensions, spaces satisfying condition (25) coincide with spaces of symmetric symplectic connection. For 2 dimensions we obtain at once $T_{ijk} = 0$. We begin the consideration of the case of 4 dimensions with a locally flat space, which can be given by a tensor a_{ij} satisfying the relations (19) and $\Gamma^l_{jk} = 0$. Since $\nabla_i \sqrt{a} = \sqrt{a}\, b_i$, in the case under consideration the condition $b_i = 0$ is equivalent to the condition $\partial a/\partial x^i = 0$, i.e.

$$\det \|a_{ij}\| = \text{const} \neq 0. \tag{30}$$

We have

$$\det \|a_{ij}\| = (a_{12}a_{43} + a_{23}a_{41} + a_{31}a_{42})^2 \quad (i, j = 1, 2, 3, 4). \tag{31}$$

Substituting here the values of a_{ij} from (19), we obtain the square of a polynomial linear with respect to x^i, since the remaining terms vanish.

In order to satisfy condition (30), it is necessary and sufficient to require that the coefficients of the unknowns x^i of this polynomial be equal to zero and that the determinant of the matrix $\|\beta_{ij}\|$ be nonzero. But it is easy to see that these conditions can be satisfied only in the case $\alpha_{ijk} = 0$, since setting equal to zero the coefficients of x^i leads to the system

$$\begin{aligned} 0 + \beta_{12}(-\alpha_{134}) + \beta_{13}(\alpha_{124}) + \beta_{14}(-\alpha_{123}) = 0,\\ \beta_{21}(\alpha_{234}) + 0 + \beta_{23}(\alpha_{124}) + \beta_{24}(-\alpha_{123}) = 0,\\ \beta_{31}(\alpha_{234}) + \beta_{32}(-\alpha_{134}) + 0 + \beta_{34}(-\alpha_{123}) = 0,\\ \beta_{41}(\alpha_{234}) + \beta_{42}(-\alpha_{134}) + \beta_{43}(\alpha_{124}) + 0 = 0. \end{aligned} \tag{32}$$

Thus, flat spaces of 4 dimensions with invariant volume are symplectic.

Further, a curved space of 4 dimensions with invariant volume will be a space of symplectic connection, since otherwise the flat space tangent to this space at some point would not be symplectic.

Let now the dimension of the space be $2n > 4$. Consider again a locally flat space. In this case condition (30) can be satisfied if one sets: 1) $\alpha_{ijk} = 0$, if at least one of the indices i, j, k is greater than n; 2) $\beta_{ij} = 0$, if both indices are greater than n.

Since for $n > 2$ not all $\alpha_{ijk} = 0$, we directly obtain a locally flat space with invariant volume, distinct from a symplectic one, for the condition $|\beta_{ij}| \neq 0$, and consequently $|a_{ij}| \neq 0$, can always be fulfilled.

The existence of curved spaces with $b_i = 0$, distinct from spaces of symplectic connection in the case $n > 2$, follows directly from this, since the components of the object γ_{ijk} do not enter into the components b_i and, consequently, may be chosen arbitrarily.

Moscow City Pedagogical Institute
named after V. P. Potemkin

Received
8 VI 1957

CITED LITERATURE

1. V. G. Lemlein, DAN, **115** No. 4 (1957).

Note: Figure translations are in progress. See original paper for figures.