ON THE REPRESENTATION OF NUMBERS IN THE FORM OF THE SUM OF A PRIME NUMBER AND A POWER OF A GIVEN INTEGER

n=p+a^i,

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Abstract

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MATHEMATICS

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ON THE REPRESENTATION OF NUMBERS IN THE FORM OF THE SUM OF A PRIME NUMBER AND A POWER OF A GIVEN INTEGER

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§ 1. The present paper is devoted to the study of the question of the number of representations of natural numbers n in the form

$$n = p + a^i,$$

where p is a prime number; $a \geqslant 2$ is a given integer; $i \geqslant 0$ is an integer.

Theorem 1. *In the interval $(0, x)$ there are more than $\frac{\alpha x}{\lg a}$ numbers representable in one and only one way in the form of the sum of a prime number and a power of the given integer $a > a_0$, where α is an absolute positive constant.*

Without a restriction on the number a, Theorem 2 is valid.

Theorem 2. *There exists a constant number k, independent of x and a, such that the number of numbers $n \leqslant x$ for which the equation $n = p + a^i$, where p is prime, $a \geqslant 2$ is a given integer, $i \geqslant 0$ is an integer, has $1, 2, \dots, k$ solutions, will be more than $\frac{\gamma x}{\lg a}$, where $\gamma > 0$ is a constant.*

Theorems 1 and 2 refine known results of N. P. Romanov ([1]), E. Landau ([3]), and are obtained from a more general theorem.

Theorem 3. *Let $\psi(n, x)$ be the number of solutions of the equation*

$$n = p + a^i,$$

where $p \leqslant x$ is a prime number, $a \geqslant 2$ is a given integer, $i \geqslant 0$ is an integer, $a^i \leqslant x$. Further, let $F_m(x)$ be the number of numbers $n \leqslant 2x$ for which $\psi(n, x) = m$, $m > 0$ an integer; let k be any odd positive number. Put also

$$\Phi_k(x) = F_1(x) + \dots + F_k(x).$$

Then

$$\Phi_k(x) \geqslant \frac{4kx}{(k+1)^2 \lg a}\left(1 - \frac{c_1 \lg^3 \lg 2a}{k \lg a} - \frac{c_2}{\lg x}\right),$$

where c_1, c_2 are positive absolute constants.

We outline the proof of Theorem 3. We have

$$\sum_{m=1}^{k} mF_m(x) = \sum_{n=1}^{2x} \psi(n,x) - \sum_{\substack{n=1 \\ \psi(n,x)>k}}^{2x} \psi(n,x). \tag{1}$$

Next, we note that the subtracted term in equality (1) does not exceed

$$\sum_{n=1}^{2x} \psi(n,x)[\psi(n,x)-k] + \sum_{m=1}^{k-1} m(k-m)F_m(x). \tag{2}$$

From (1) and (2) it follows that

$$\max_{1\le m<k}\{m(k-m+1)\}\Phi_k(x) \ge \sum_{n=1}^{2x} \psi(n,x)[k+1-\psi(n,x)]. \tag{3}$$

Now we shall use the identity of N. P. Romanov [1]

$$\sum_{n=1}^{2x} \psi^2(n,x) = \sum_{n=1}^{2x} \psi(n,x) + 2\sum_{n=1}^{x} A_1(n,x)A_2(n,x), \tag{4}$$

where $A_1(n,x)$ and $A_2(n,x)$ are, respectively, the numbers of solutions of the equations:

$$p_i - p_j = n, \qquad p_i, p_j \le x \text{ —prime numbers;}$$

$$a^u - a^t = n, \qquad a^u, a^t \le x, \qquad u, t \ge 0 \text{ —integers.}$$

From (3) and (4) we obtain

$$\Phi_k(x) \ge \frac{4k}{(k+1)^2}\sum_{n=1}^{2x} \psi(n,x) - \frac{8}{(k+1)^2}\sum_{n=1}^{x} A_1(n,x)A_2(n,x). \tag{5}$$

It is not difficult to see that

$$\sum_{n=1}^{2x} \psi(n, x) = \pi(x) N(x), \tag{6}$$

where $\pi(x)$ is the number of primes $\leq x$; $N(x)$ is the number of numbers $a^i \leq x$.

On the basis of the investigations of Viggo Brun, L. G. Schnirelmann [2], N. P. Romanov [1], and E. Landau,

$$\sum_{n=1}^{x} A_1(n, x) A_2(n, x) < cx \frac{\lg^3 \lg 2a}{\lg^2 a}, \tag{7}$$

where $c > 0$ is an absolute constant.

Combining the estimates (5)–(7), we obtain Theorem 3.

§ 2. The following two propositions are a supplement to Theorems 1 and 2.

Theorem 4. There exists an infinite set of numbers n for which (in the notation of Theorem 3), as $x \to \infty$, we have

$$\psi(n, x) > \delta \lg \lg n,$$

where $\delta > 0$ is some constant.

Theorem 5. If $\gamma(x)$ is any positive function, increasing without bound as $x \to \infty$, and $M(x)$ is the number of numbers $n \leq x$ for which

$$\psi(n, x) > \gamma(x),$$

then

$$M(x) = o(x).$$

Tashkent State Pedagogical Institute
named after Nizami

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CITED LITERATURE

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[4] K. Prachar, *J. London Math. Soc.*, **29**, No. 3 (1954).