MATHEMATICS

1957

SovietRxiv

Abstract

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MATHEMATICS

I. I. PYATETSKII-SHAPIRO

ON AN ESTIMATE OF THE DIMENSION OF THE SPACE OF AUTOMORPHIC FORMS FOR CERTAIN TYPES OF DISCRETE GROUPS

(Presented by Academician A. N. Kolmogorov, 30 X 1956)

Let D be a bounded domain in n-dimensional complex space; let Γ be a discrete group of analytic automorphisms. If the fundamental domain D/Γ is compact, then the totality of automorphic functions forms a field of algebraic functions of n independent variables (^{1}). This assertion also remains valid for certain discrete groups with noncompact fundamental domain, for example for the modular groups of Siegel and Hilbert and for the Hermitian modular group introduced by Braun. As is known, for its proof it is sufficient to show that the dimension of the space of automorphic forms of weight m, as $m \to \infty$, grows no faster than cm^n, where n is the complex dimension of the domain of definition of the automorphic forms, and c is a certain constant (^{1}).

In the present work a general class of discrete groups in bounded symmetric domains is described for which the required estimate holds.

Siegel, and after him other authors, used for estimating the dimension of the space of automorphic forms an analytic mapping of a bounded symmetric domain onto a certain unbounded domain, analogous to the upper half-plane in the case of one variable, and Fourier series. We shall consider two sorts of domains playing the role of upper half-planes, called in the present note Siegel domains of the first and second kinds. In Siegel domains of the first kind Fourier series are used; in Siegel domains of the second kind Fourier series are used whose coefficients are Jacobi functions.

We now pass to the definition of Siegel domains of the first kind.

Definition 1. Let V be an open convex cone in n-dimensional Euclidean space, not containing all points of any one straight line. We shall agree to call the cylindrical domain H with base V a **Siegel domain of the first kind**.

Let us note some general properties of Siegel domains of the first kind.

1) Every Siegel domain of the first kind is analytically equivalent to a bounded domain.

2) Linear transformations of a Siegel domain of the first kind into itself have the form

$$z \to Lz + b,$$

where L is an affine transformation of the cone V into itself; b is a real vector.

3) Let Γ be a discrete group of linear transformations of H into itself; let Γ_0 be a subgroup of Γ consisting of parallel translations. We shall call Γ a **normal discrete group** if Γ_0 has n generators.

For points of n-dimensional Euclidean space one may introduce the notion of "greater" with the aid of the cone V. We shall agree to say that $a > b$, if $a-b \in V$. Denote now by H_a the totality of all points $z = x + iy$ for which $y > a$.

Lemma 1. Let a, b be points of n-dimensional Euclidean space, $b > a > 0$; let M be some positive constant; and let E_M be a linear space of functions, analytic in H and automorphic with respect to the normal discrete group Γ, such that for every function $f \in E_M$ the inequality

$$\max_{z \in H_a} |f(z)| < M \max_{z \in H_b} |f(z)|. \tag{1}$$

holds. Then the dimension of E_M does not exceed

$$c(\ln M)^n, \tag{2}$$

where c depends only on V, a, b, Γ, and does not depend on M.

Let us give some examples of Siegel domains of the first kind:

1) The set of all complex square matrices Z for which*

$$\frac{Z - \overline{Z}'}{i} > 0.$$

2) The set of all complex skew-symmetric matrices Z of even order, for which

$$ZJ + J\overline{Z}' > 0.$$

3) The set of all complex symmetric matrices $Z = X + iY$, for which $Y > 0$.

4) The set of all points $z = x + iy$ of n-dimensional complex space for which

$$y_1 > \sqrt{y_2^2 + \cdots + y_n^2}, \qquad y = (y_1, y_2, \ldots, y_n).$$

The domains considered in these four examples are analytically equivalent to irreducible bounded symmetric homogeneous domains. Unfortunately, not every bounded symmetric domain has an equivalent Siegel domain of the first kind. We shall now introduce one more class of unbounded domains (Siegel domains of the second kind) and show that every irreducible symmetric domain** is analytically equivalent either to a Siegel domain of the first kind, or to a Siegel domain of the second kind.

Definition 2. Let V be a cone in n-dimensional Euclidean space; let $F(u, v)$ be a function of a pair of vectors u and v of m-dimensional complex space, whose value is an n-dimensional vector. We shall agree to call $F(u, v)$ V-Hermitian if the following conditions are satisfied:

1) $F(u, v) = \overline{F(v, u)}$.
2) $F(\lambda u_1 + \mu u_2, v) = \lambda F(u_1, v) + \mu F(u_2, v)$.
3) $F(u, u) \in \overline{V}$ ($\overline{V}$ is the closure of the cone V).
4) The convex hull spanned by the vectors $F(u, u)$ coincides with $\overline{V}$.
5) $F(u, u) = 0$ only if $u = 0$.

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* A Hermitian matrix $A > 0$ if all its eigenvalues are positive.

** As is known, there exist 6 types of irreducible bounded symmetric domains, of which 2 are exceptional. In the present note only the non-exceptional types of domains are considered.

Definition 3. By a Siegel domain of the second kind of characteristic (n, m) we shall mean the set of all points (z, u) of an $(n + m)$-dimensional complex space for which

$$\operatorname{Im} z - F(u, u) > 0.$$

Let us note the following properties of Siegel domains of the second kind:

1) Every Siegel domain of the second kind is analytically equivalent to a bounded domain.
2) A Siegel domain of the second kind admits the group of transformations

$$(z, u) \to (z + a + 2iF(u, c) + iF(c, c), u + c), \tag{3}$$

where a is an arbitrary real vector and c is an arbitrary complex vector.

3) A Siegel domain of the second kind is a skew product whose base is an m-dimensional complex space and whose fiber is a Siegel domain of the first kind.

4) Let D be an irreducible bounded symmetric domain in an N-dimensional complex space; let D_0 be the part of the boundary of D, invariant under analytic automorphisms of D, of minimal dimension; let ν be the real dimension of D_0. Always $\nu \geq N$. If $\nu = N$, then D is analytically equivalent to some Siegel domain of the first kind; if $\nu > N$, then D is analytically equivalent to a Siegel domain of the second kind of characteristic $(2N-\nu, \nu-N)$.

5) Let Γ be a discrete subgroup of the group of transformations of the form (3). We shall call Γ a normal discrete group if its commutant and the factor group by the commutant have, respectively, n and $2m$ generators. Denote by H_a the set of (z,u) for which $\operatorname{Im} z - F(u,u) > a$.

Lemma 2. Any analytic function automorphic in H with respect to a normal discrete group Γ is bounded in H_a, $a > 0$.

Let E be a linear space of analytic functions automorphic with respect to Γ, for which the inequality

$$\max_{z \in H_a} |f(z)| < M \max_{z \in H_b} |f(z)|, \qquad a < b.$$

holds. The dimension of E does not exceed $c(\ln M)^{n+m}$, where c is a certain constant.

6) We give some examples of Siegel domains of the second kind.

a) Let Z be a square matrix of order r; U a rectangular matrix with r rows and k columns. The set of all (Z,U) for which

$$\frac{1}{2i}(Z - \overline{Z}') - U\overline{U}' > 0,$$

forms a Siegel domain of the second kind.

b) Let Z be a skew-symmetric matrix of order $2t$; U a rectangular matrix with $2t$ rows and 2 columns, satisfying the condition

$$J_t \overline{U} = U J_1,$$

where

$$J_t = \begin{pmatrix} 0 & E_t \\ -E_t & 0 \end{pmatrix},$$

and E_t is the identity matrix of order t. The set of all (Z,U) for which

$$ZI + J\overline{Z} - U\overline{U}' > 0,$$

forms a Siegel domain of the second kind.

It is obvious that every bounded symmetric domain is equivalent either to a Siegel domain of the first kind, or to a Siegel domain of the second kind, or to a product of Siegel domains of the first and second kinds.

In what follows, when speaking of a "Siegel domain," we shall mean either a Siegel domain of the first kind, or a Siegel domain of the second kind, or a product of Siegel domains of the first and second kinds.

Domains of the form H_a, where H is some Siegel domain ($a > 0$), shall be called **neighborhoods of an infinitely distant point**.

Let D be some bounded domain in N-dimensional complex space; let z be some point of the boundary of D; let φ be an analytic mapping of D onto a Siegel domain that sends z to an infinitely distant point; and let Γ be a discrete group of analytic automorphisms of D. We shall call z a **normal parabolic point** for the group Γ if, under the above-mentioned mapping, Γ is transformed into a group containing a normal discrete subgroup. Let F be some generalized fundamental domain for Γ. Suppose that there exists a finite number of normal parabolic points $z_1, \dots, z_k$ with neighborhoods $S_1, \dots, S_k$, such that

$$F - \sum_{\nu=1}^{k} F_\nu$$

is contained strictly inside F, where $F_\nu = S_\nu \cap F$. A discrete group possessing a generalized fundamental domain with the properties indicated above will be called a **discrete group of regular structure**.

Definition 4. A function $f(z)$, regular in D, is called an **automorphic form** if:

1) $f(\gamma z) = j_\gamma^{-m}(z) f(z)$.
2) Let z_0 be a normal parabolic point. The ratio

$$\left| \frac{f(z)}{j_{\varphi_0}^m(z)} \right|$$

is bounded in any neighborhood of z_0.

We now formulate the main theorem of the present note.

Theorem. *Let D be a bounded domain in N-dimensional complex space; let Γ be a discrete group of analytic automorphisms of D of regular structure. The dimension of the space of automorphic forms of weight m does not exceed cm^N, where c does not depend on m and depends only on D and Γ.*

In conclusion I express my gratitude to A. O. Gelfand and I. R. Shafarevich for their attention to the present work.

Kaluga State
Pedagogical Institute

Received
15 X 1956

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Note: Figure translations are in progress. See original paper for figures.

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