ON THE ε-ENTROPY OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract

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MATHEMATICS

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ON THE ε-ENTROPY OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Considering various classes F of analytic functions $f(z)$, we shall be especially interested in their behavior on the segment of the real axis $\Delta_T = \{z;\ -T \le z \le +T\}$ (always assuming that it lies in the domain of definition of $f \in F$) and, accordingly, shall use the metric $\rho_T(f_1, f_2) = \max |f_1(z) - f_2(z)|, \quad z \in \Delta_T$.

As in the note of A. N. Kolmogorov ([1]), we shall denote by $N_\varepsilon^T(F)$ the minimal number of elements of an ε-covering of F, i.e. a covering of F by sets whose diameter in the metric ρ_T is $\le \varepsilon$. The binary logarithm $\log N_\varepsilon^T(F) = H_\varepsilon^T(F)$ will be called the ε-entropy of the class F on the segment Δ_T. It seems to us essential, along with $N_\varepsilon^T(F)$, to consider the quantity $R_\varepsilon^T(F)$, equal to the maximal number of elements of an ε-distinguishable subset of F, i.e. a set contained in F for any two elements f_1 and f_2 of which $\rho_T(f_1, f_2) > \varepsilon$. $\log_2 R_\varepsilon^T(F) = C_\varepsilon^T(F)$ will be called the ε-capacity of the class F on the segment Δ_T.

It is easy to show that

$$C_\varepsilon^T(F) \le H_\varepsilon^T(F). \tag{1}$$

We shall consider the following classes of analytic functions:
$A_h^T(M)$–the class of functions analytic and bounded by the constant M in the domain G_h^T of points z of the form $z = t + u,\ t \in \Delta_T,\ |u| \le h$;

$F_{s,\sigma}^T(M)$–the class of entire functions satisfying, for every t from the segment Δ_T and every u, the inequality $|f(t+u)| \le Me^{\sigma|u|^s},\ s \ge 1$;

$B_\sigma(M)$–the class of entire functions satisfying the relation $|f(z)| \le Me^{\sigma|\operatorname{Im} z|}$. The class $B_\sigma(M)$ would naturally be denoted by $F_{1\sigma}^\infty(M)$, since it coincides with the intersection of all classes $F_{1\sigma}^T(M)$ with finite T.

In formulating the results we shall use Bourbaki' s notation $\asymp$ and $\sim$ for strong and weak equivalence, explained in ([1]). In some development of these, we shall say that $f(\varepsilon, T)$ and $g(\varepsilon, T)$ are weakly equivalent as $\varepsilon \to 0$ under one or another condition S on ε and T, uniformly in T, if there exist constants $0 < a \le A$ and

$\varepsilon \leq \varepsilon_0$ such that
$a \leq f(\varepsilon, T)/g(\varepsilon, T) \leq A$ for $\varepsilon \leq \varepsilon_0$ and pairs (ε, T) satisfying the condition S.

Theorem 1*.

$$\frac{2\sigma}{\pi} \log \frac{1}{\varepsilon} \asymp \liminf_{T\to\infty} \frac{1}{2T} H_\varepsilon^T(B_\sigma(M)) \asymp \limsup_{T\to\infty} \frac{1}{2T} H_\varepsilon^T(B_\sigma(M)).$$

* Theorems 1-3 remain unchanged also for the functions C_ε^T.

Theorem 2. Uniformly in $T \geqslant 0$,

$$H_\varepsilon^T\left(A_h^T(M)\right) \asymp \left(\frac{\log(1/\varepsilon)}{\log(1/T+1)} + 1\right) \log \frac{1}{\varepsilon}.$$

Theorem 3. For $s \geqslant 1$, uniformly in $T \geqslant 0$,

$$H_\varepsilon^T\left(F_{s,\sigma}^T(M)\right) \asymp \left(\frac{\log(1/\varepsilon)}{\log\left(\frac{(\log(1/\varepsilon))^{1/\alpha}}{T} + 1\right)} + 1\right) \log \frac{1}{\varepsilon}.$$

The right-hand sides of the equivalences in Theorems 2 and 3 for $T = 0$ are taken to be equal to $\log(1/\varepsilon)$.

We note that the class $B_\sigma(M)$ in the metric ρ_T (for any T) is the closure of the class $B'_\sigma(M)$ of finite sums

$$f(z) = \sum_{k=-n}^{+n} c_k e^{i\lambda_k z},$$

satisfying on the real axis the condition $|f| \leqslant M$, and with frequencies λ_k from the interval $-\sigma \leqslant \lambda \leqslant +\sigma$ (([2], pp. 160-164). Therefore Theorem 1 may be regarded as one of the justifications of the idea (apparently first advanced by V. A. Kotelnikov ([3]) and widely accepted in information theory) that the amount of information contained in specifying, on an interval of length τ, a function with spectrum bounded by a frequency band of width 2σ, for large τ, is equivalent to the amount of information in specifying $\frac{2\sigma}{\pi}\tau$ real numbers.

We also note some weak equivalences following from Theorems 2 and 3, **valid uniformly in** T, under various restrictions:

$$2_a. \qquad H_\varepsilon^T\left(A_h^T(M)\right) \asymp \log \frac{1}{\varepsilon} \left(1 + \frac{\log \varepsilon}{\log T}\right) \quad \text{for } T \leqslant T_0.$$

$$2_b. \qquad H_\varepsilon^T\left(A_h^T(M)\right) \asymp \log^2 \frac{1}{\varepsilon} \quad \text{for } 0 < T_0 \leqslant T \leqslant T_1.$$

$$2_c. \qquad H_\varepsilon^T\left(A_h^T(M)\right) \asymp T \left(\log \frac{1}{\varepsilon}\right)^2 \quad \text{for } 0 < T_0 \leqslant T.$$

$$3_a. \qquad H_\varepsilon^T\left(F_{s\sigma}^T(M)\right) \asymp \log \frac{1}{\varepsilon}\left(1 + \frac{\log \varepsilon}{\log T}\right) \quad \text{for } T \leqslant \frac{c}{(\log(1/\varepsilon))^\alpha}, \quad c > 0, \quad \alpha > \frac{1}{s}.$$

$$3_b. \qquad H_\varepsilon^T\left(F_{s\sigma}^T(M)\right) \asymp \frac{(\log(1/\varepsilon))^2}{\log\log(1/\varepsilon)} \quad \text{for } \frac{c}{(\log(1/\varepsilon))^\alpha} \leqslant T \leqslant c'\left(\log \frac{1}{\varepsilon}\right)^\beta,$$

$$\alpha > 0, \quad \beta < \frac{1}{s}, \quad c,\ c' > 0.$$

$$3_c. \qquad H_\varepsilon^T\left(F_{s\sigma}^T(M)\right) \asymp T \log^{2-1/s}\left(\frac{1}{\varepsilon}\right) \quad \text{for } T \geqslant c_1\left(\log \frac{1}{\varepsilon}\right)^\beta, \quad \beta \geqslant \frac{1}{s}, \quad c_1 > 0.$$

Relations 2_b and 3_b are valid, in particular, for constant $T > 0$ (for constant T, the relations 2_b are indicated in ([1])); their comparison shows that for constant T the passage from the class A_h to the seemingly much narrower class $F_{s\sigma}$ has little effect on the order of growth of H_ε.

From 2_c and 3_c follow the relations

$$2. \qquad \liminf_{T\to\infty} \frac{1}{2T} H_\varepsilon^T\left(A_h^T(M)\right) \asymp \limsup_{T\to\infty} \frac{1}{2T} H_\varepsilon^T\left(A_h^T(M)\right) \asymp \left(\log \frac{1}{\varepsilon}\right)^2.$$

$$3. \qquad \liminf_{T\to\infty} \frac{1}{2T} H_\varepsilon^T\left(F_{s,\sigma}^T(M)\right) \asymp \limsup_{T\to\infty} \frac{1}{2T} H_\varepsilon^T\left(F_{s\sigma}^T(M)\right) \asymp \left(\log \frac{1}{\varepsilon}\right)^{2-1/s}.$$

We note that 2 and 3 are preserved when passing to the classes A_h^∞ and $F_{s\sigma}^\infty$. It is natural to compare them with Theorem 1.

Proof of Theorem 1. a) **Upper estimate for the quantity** $H_\varepsilon(B_\sigma^T(M))$. It is known (see, for example, (4), p. 269) that for every function $g(z) \in B_{\sigma'}(M)$, $\sigma' < \pi$, the following interpolation formula holds:

$$g(z)=\frac{\sin\pi z}{\pi\omega}\sum_{k=-\infty}^{+\infty}(-1)^k\frac{g(k)\sin\omega(k-z)}{(k-z)^2}, \tag{2}$$

where $0<\omega<\pi-\sigma'$.

Obviously, if $f(z)\in B_\sigma(M)$, then $g(z)=f\left(\frac{\sigma'}{\sigma}z\right)\in B_{\sigma'}(M)$, whence from (2) we obtain, for $f(z)\in B_\sigma(M)$,

$$f(z)=\frac{\sin\frac{\pi\sigma}{\sigma'}z}{\pi\omega}\sum_{k=-\infty}^{+\infty}(-1)^k\frac{f\left(k\frac{\sigma'}{\sigma}\right)\sin\frac{\omega\sigma}{\sigma'}\left(k\frac{\sigma'}{\sigma}-z\right)}{\left(\frac{\sigma}{\sigma'}\right)^2\left(k\frac{\sigma'}{\sigma}-z\right)^2},\qquad \sigma'<\pi,\quad 0<\omega<\pi-\sigma'.$$

Fix ω. Choose $n=n(\varepsilon,\omega)$ so that $\sum_{|s|>n}\frac{1}{s^2}<\frac{\pi}{M}\varepsilon\omega$. Then for $t\in{\Delta_T}^*$

$$\left|\frac{\sin\frac{\pi\sigma}{\sigma'}t}{\pi\omega}\sum_{|k|>n+[T\sigma/\sigma']+1}\frac{f\left(k\frac{\sigma'}{\sigma}\right)\sin\frac{\omega\sigma}{\sigma'}\left(k\frac{\sigma'}{\sigma}-t\right)}{\left(\frac{\sigma}{\sigma'}\right)^2\left(k\frac{\sigma'}{\sigma}-t\right)^2}\right|\le\frac{M}{\pi\omega}\sum_{|s|>n}\frac{1}{s^2}<\varepsilon. \tag{3}$$

We approximate $f(z)$ by a function $\tilde f(z)$ of the form

$$\tilde f(z)=\frac{\sin\frac{\pi\sigma}{\sigma'}z}{\pi\omega}\,\varepsilon\sum_{|k|<n+[T\sigma/\sigma']+1}(-1)^k\frac{\left(\left[\frac{\operatorname{Re}f\left(k\frac{\sigma'}{\sigma}\right)}{\varepsilon}\right]+i\left[\frac{\operatorname{Im}f\left(k\frac{\sigma'}{\sigma}\right)}{\varepsilon}\right]\right)\sin\frac{\omega\sigma}{\sigma'}\left(k\frac{\sigma'}{\sigma}-z\right)}{\left(\frac{\sigma}{\sigma'}\right)^2\left(k\frac{\sigma'}{\sigma}-z\right)^2}.$$

It is not hard to show that

$$\rho_T(f,\tilde f)\equiv\max_{t\in\Delta_T}|f(t)-\tilde f(t)|\le C(\omega)\varepsilon,$$

where $C(\omega)$ is a constant depending only on ω.

For each constructed function $\tilde f$, consider the set $\{U_{\tilde f}\}$ of functions $f(z)$ such that $\rho_T(f,\tilde f)\le C(\omega)\varepsilon$. It is easy to see that $\{U_{\tilde f}\}$ forms a $2C(\omega)\varepsilon$-covering of the space $B_\sigma^T(M)$. The number $N_{\varepsilon T}$ of sets in this covering satisfies the inequality

$$\overline{N}_{\varepsilon T}\le(2[M/\varepsilon]+1)^{2(2[T\sigma/\sigma']+2n+3)},$$

whence

$$H^T_{2C(\omega)\varepsilon}(B^T_\sigma(M)) \leq (4[T\sigma/\sigma'] + n')\log(2[M/\varepsilon]+1),$$

where

$*[A]$ is the integer part of A.

where $n' = 4n+6$, whence

$$H^T_\varepsilon(B_\sigma(M)) \ll (4[T\sigma'/\sigma'] + n')\log(2[2MC(\omega)/\varepsilon]+1). \quad (4)$$

From (4) it follows immediately that

$$\lim_{\varepsilon\to 0}\overline{\lim_{T\to\infty}}\frac{H^T_\varepsilon(B_\sigma(M))}{2T\log(1/\varepsilon)} \ll \frac{2\sigma}{\sigma'} \ll \frac{2\sigma}{\pi},$$

since σ' is an arbitrary number less than π.

b) **A lower estimate for the quantity** $C^T_\varepsilon(B_\sigma(M))$**.** Let $a>0$, $\sigma' = \sigma - a$, $n = [T\sigma'/\pi]$, and let $K = \{(k_{-n}, k'_{-n}), \dots, (k_0, k'_0), \dots, (k_n, k'_n)\}$ be a set of integers, each of which does not exceed $[M'/\varepsilon]$ (we shall take care of M' later).

Consider the set of functions $\{f_K(z)\}$, where

$$f_K(z) = \varepsilon \sum_{|j|<n} \frac{(k_j + ik'_j)\sin\sigma'(z - j\pi/\sigma')\sin a(z - j\pi/\sigma')}{a\sigma'(z - j\pi/\sigma')^2}.$$

It is easy to prove that all $f_K(t)$ are bounded by one and the same constant $M'C(\sigma')$, independent of n. Put $M' = M/C(\sigma')$. Moreover, $f_K(z)$ is of exponential growth with exponent $\sigma' + a = \sigma$; consequently (([2]), p. 151), they all belong to $B_\sigma(M)$.

The set of functions $f_K(z)$ is ε-distinguishable. The number $N_{\varepsilon T}$ of these functions is equal to

$$(2[M/C\sqrt{2}\varepsilon]+1)^{2(2[T\sigma'/\pi]+1)},$$

whence

$$C^T_\varepsilon(B_\sigma(M)) \gg \log N_{\varepsilon T} = (4[(\sigma-a)T/\pi]+2)\log(2[M/C\sqrt{2}\varepsilon]+1). \quad (5)$$

From (5) one may obtain that

$$\lim_{\varepsilon\to 0}\lim_{T\to\infty}\frac{C_\varepsilon^T(B_\sigma(M))}{2T\log(1/\varepsilon)} \gg \frac{2(\sigma-a)}{\pi} \gg \frac{2\sigma}{\pi},$$

since a is arbitrary.

Using (1), we finally obtain

$$\liminf_{T\to\infty}\frac{H_\varepsilon^T(B_\sigma(M))}{2T} \sim \limsup_{T\to\infty}\frac{H_\varepsilon^T(B_\sigma(M))}{2T} \sim \frac{2\sigma}{\pi}\log\frac{1}{\varepsilon}.$$

The scheme of proof of Theorems 2 and 3 is the same as that of Theorem 1. The upper estimates are obtained by approximating the coefficients of Taylor series; the lower estimates for small T (in Theorem 2 for $T \ll \lambda h$, in Theorem 3 for $T \ll \lambda(\log(1/\varepsilon))^{1/s}$, where λ is a certain number) are obtained by considering interpolation polynomials, and for large T by trigonometric interpolation.

In conclusion I consider it my pleasant duty to express gratitude to A. N. Kolmogorov, who posed the problems considered here and devoted much attention to the work.

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CITED LITERATURE

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[2] N. I. Akhiezer, *Lectures on Approximation Theory*, 1947.
[3] V. A. Kotelnikov, *Materials for the 1st All-Union Congress on the Problem of Communication Reconstruction*.
[4] B. Ya. Levin, *Distribution of Zeros of Entire Functions*, 1956.

Note: Figure translations are in progress. See original paper for figures.