Reports of the Academy of Sciences of the USSR

1957

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR
1957. Volume 116, No. 5

MATHEMATICS

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ON THE QUESTION OF SUMMATION OF FUNCTIONS OF THE CLASSES $B^{(r)}$

(Presented by Academician A. N. Kolmogorov, 25 IV 1957)

1. A. N. Kolmogorov (^{1}) investigated the behavior of the upper bounds of the remainders of the Fourier series of functions of the classes $W^{(r)}$ $(r = 1, 2, \ldots)$. An analogous problem for the classes $B^{(r)}$ $(r = 1, 2, \ldots)$ was considered by S. B. Stechkin (^{2}). In the present note a theorem is proved which generalizes S. B. Stechkin's results to a comparatively broad class of summation methods with a triangular Toeplitz matrix, and an application to the theory of best approximations is indicated.

By $B^{(r)}$ $(r = 1, 2, \ldots)$ one denotes the class of functions

$$f(z) = \sum_{k=0}^{\infty} c_k z^k,$$

analytic in the disk $|z| < 1$ and satisfying there the condition $|f^{(r)}(z)| \leqslant 1$, where $f^{(r)}(z)$ is the r-th derivative of the function $f(z)$. Denote by $P_n(z, f^{(r)})$ the Taylor sum, and by $\sigma_n(z, f^{(r)})$ the $(C, 1)$-mean of order n of the series $f^{(r)}(z)$; then for $f(z) \in B^{(r)}$ the well-known relations hold

$$P_n(z, f^{(r)}) = O[\ln(n+1)], \qquad |\sigma_n(z, f^{(r)})| \leqslant 1 \quad (n = 0, 1, 2, \ldots) \tag{1}$$

uniformly with respect to z for $|z| \leqslant 1$ and with respect to all functions of the class $B^{(r)}$.

Let us now single out classes of summation methods. Denote by T_m $(m = 1, 2, \ldots)$ the class of regular summation methods with multipliers $\mu_n(k) = \varphi_1(k)$ for $0 \leqslant k \leqslant n_1$; $\mu_n(k) = \varphi_2(k)$ for $n < k \leqslant n_2$; ...; $\mu_n(k) = \varphi_m(k)$ for $n_{m-1} < k \leqslant n$; $\mu_n(n+1) = 0$. Here $\varphi_1, \varphi_2, \ldots, \varphi_m$ are functions of the integer arguments n, for* $n \geqslant n_0$, and k, for $k = 0, 1, 2, \ldots$, which ensure the regularity of the summation method defined by them; moreover, each of them, as a function of k (for fixed n), can be written in the form of a single analytic expression for all k in the domain of definition; otherwise these functions are arbitrary.

The Taylor sums, (C, m)- and (H, m)-means, the Vallée-Poussin integral, etc., belong to the class T_1. Jackson' s method and the Vallée-Poussin sums are representatives of the class T_2.

2. Let a summation method $\mu \in T_1$ be given by multipliers $\mu_n(k)$ $(0 \leqslant k \leqslant n)$, which we represent in the form of a Newton interpolation polynomial with remainder

$$\mu_n(k) = \sum_{j=0}^{p} \beta_j k \ldots (k+1-j) + \rho_p(n,k) \quad (0 \leqslant k \leqslant n). \tag{2}$$

For a function $f(z) \in B^{(r)}$ we construct the μ-mean

$$A_n(z, f; \mu) = \sum_{k=0}^{n} \mu_n(k) c_k z^k \quad (n = 0, 1, 2, \ldots). \tag{3}$$

* The number n_0 is an arbitrary nonnegative number.

For $A_n(z, f; \mu)$ the following theorem is valid.

Theorem. For any function $f(z) \in B^{(r)}$ $(r = 1, 2, \ldots)$ and any summation method $\mu \in T_1$,

$$A_n(z, f; \mu) = \sum_{j=0}^{r-1} \beta_j z^j f^{(j)}(z) + B_n z^r \rho_{n-r}(z, f^{(r)}) + D_n \tag{4}$$

$$(n = r-1, r, r+1, \ldots)$$

uniformly with respect to z for $|z| \leq 1$ and for each of the classes $B^{(r)}$ and T_1, where $f^{(j)}(z)$ is the j-th derivative of the function $f(z)$;

$$B_n = \sum_{j=0}^{r-1} \frac{\beta_j}{(n-r+2)\ldots(n+1-j)} + \beta_r + \sum_{j=r+1}^{p} \beta_j (n+1)\ldots(n+j-r),$$

$$D_n = \sum_{j=0}^{r-1} \beta_j \frac{2(r-j)+1}{(n-r+3)\ldots(n+2-j)} \theta_n(j) +$$

$$+ \sum_{j=r+1}^{p} \left\{ (2^{j-r} - 1)\theta_n(j-r) + O\left[\frac{\ln(n-r+2)}{n+2}\right] \right\} \beta_j (n+1)\ldots(n+j-r) +$$

$$+ \sum_{k=0}^{n} c_k z^k \rho_p(n,k)$$

$$(|\theta_n(j)| \leq 1; \quad 0 \leq j \leq p; \quad n \geq r-1; \quad r = 1, 2, \dots).$$

The special cases of the theorem for Taylor sums and Fejér means were proved by S. B. Stechkin (^{2}).

The proof of the theorem is based on two lemmas for numerical series. Consider the numerical series

$$\sum_{k=0}^{\infty} u_k, \qquad \sum_{k=0}^{\infty} (k+1) \dots (k+j) u_{k+j} \quad (j = 1, 2, \dots, r). \tag{5}$$

We shall denote their partial sums of order m by $t_m^{(j)}$, the remainders of the corresponding series by $R_m^{(j)}$, and the $(C, 1)$-means $t_m^{(r)}$ by σ_m. For the series (5) the following lemmas are valid.

Lemma 1. If $t_m^{(r)} = o(m)$, $|\sigma_m| \leq 1$ $(m = 0, 1, 2, \dots)$, then the series (5) for $j = 0, 1, \dots, r-1$ converge and

$$R_{n-j}^{(j)} = \frac{-t_{n-r}^{(r)}}{(n-r+2) \dots (n+1-j)} + \frac{2(r-j)+1}{(n-r+3) \dots (n+2-j)} \theta_n(j)$$

$$\Bigg(|\theta_n(j)| \leq 1 - \frac{r-j}{[2(r-j)+1](n-r+2)};$$

$$j = 0, 1, \dots, r-1; \quad n \geq r-1; \quad r = 1, 2, \dots).$$

From Lemma 1 there follows directly Theorem 2 of S. B. Stechkin (^{2}). For the proof of Lemma 1 we isolate in the remainder $R_{n-j}^{(j)}$ the principal term

$$\frac{-t_{n-r}^{(r)}}{(n-r+2) \dots (n+1-j)}$$

(by means of a twofold Abel transformation) and estimate quantities of higher order of smallness.

Lemma 2. Under the conditions of Lemma 1, for $j = r+1, \ r+2, \dots$ $(r = 1, 2, \dots)$ the following equalities hold:

$$\sum_{k=0}^{n} (k+1-j) \cdots k u_k =$$

$$= \left\{ t_{n-r}^{(r)} + (2^{1-r} - 1)\theta_n(j-r) + O\left[\frac{\ln(n-r+2)}{n+2}\right] \right\} (n+1) \cdots (n+j-r)$$

$$(|\theta_n(j-r)| \leqslant 1; \quad j-r = 1, 2, \dots; \quad n = 0, 1, 2, \dots).$$

Let us outline the proof of Lemma 2. As is easy to see,

$$\sum_{k=0}^{n} (k+1-j) \cdots k u_k = \sum_{k=0}^{n-r} (k+1+r-j) \cdots k a_k,$$

where $a_k = (k+1) \cdots (k+r) u_{k+r}$. To extract the principal term from the last sum and estimate the remainder, we construct, by means of the (C, m)-multipliers

$$\mu_n^{(m)}(k) = \sum_{l=0}^{m} (-1)^l C_m^l \frac{(k+1-l) \cdots (k-1)k}{(n+m+1-l) \cdots (n+m)}$$

the means

$$\sigma_{n-r}^{(j-r)} = \sum_{k=0}^{n-r} \mu_{n-r}^{(j-r)}(k) a_k$$

of the sum

$$t_{n-r}^{(r)} = \sum_{k=0}^{n-r} a_k.$$

To prove the theorem, we represent the right-hand side of equality (3), with the aid of formulas (2), in the following form:

$$A_n(z; f; \mu) =$$

$$= \left(\sum_{j=0}^{r-1} \beta_j + \beta_r + \sum_{j=r+1}^{p} \beta_j \right) \sum_{k=0}^{n} (k+1-j) \cdots k C_k z^k + \sum_{k=0}^{n} C_k z^k \rho_p(n, k). \qquad (6)$$

Now put $u_k = C_k z^k$; then

$$t_{n-r}^{(r)} = \sum_{k=0}^{n-r} (k+1) \cdots (k+r) C_{k+r} z^{k+r} = z^r p_{n-r}(z, f^{(r)}). \qquad (7)$$

From conditions (1) and equality (7) it follows that $t_{n-r}^{(r)}$ and σ_{n-r} satisfy the conditions of Lemma 1. Applying Lemmas 1 and 2 to the transformation of the sums $\sum_{j=0}^{r-1}$ and $\sum_{j=r+1}^{p}$ in the right-hand side of (6), we obtain equality (4).

3. Let us note some consequences of the theorem just proved. The theorem makes it possible to determine the order of approximation effected by a sufficiently large class T_1 of linear summation methods. It also indicates an algorithm for constructing summation methods that effect, on the classes $B^{(r)}$, approximation of the best possible order. It is easy to see that such a method is determined, in particular, by the multipliers

$$\mu_n(k) = 1 + \sum_{i=1}^{r} d_i \frac{k \cdots (k+1-i)}{(n+1)^i} + \sum_{i=r+1}^{p} d_i \frac{k \cdots (k+1-i)}{(n+1)^i} \qquad (0 \leqslant k \leqslant n)$$

under the condition that

$$1+\sum_{i=r}^{p} d_i = 0.$$

These multipliers satisfy the conditions of A. F. Timan [3] for methods that provide approximation of the order of the best possible on the classes $W^{(r)}H^{(\alpha)}$ and $\overline{W}^{(r)}H^{(\alpha)}$ for $0 \le \alpha < 1$.

Let us note one more method of constructing means that give approximation of the order of the best possible. For this purpose, from the class of summation methods T_1 we single out subclasses.

Definition. We shall say that a regular summation method with multipliers (2) belongs to the class $M_1^{(r)} \subset T_1$ $(r = 1, 2, \ldots)$, if one can indicate numbers n_0 and p_0 such that, for $n \ge n_0$ and $p \ge p_0$,

$$\beta_j = O\left(\frac{1}{n^j}\right) \quad (0 \le j \le p); \qquad \rho_p(n,k) = O\left(\frac{1}{n^r}\right) \quad (0 \le k \le n).$$

By M_1 we shall denote the subclass of the class $M_1^{(r)}$ satisfying the condition

$$\lim_{n\to\infty} n^r \cdot \rho_p(n,k) = 0 \qquad (0 \le k \le n)$$

for arbitrarily large r.

It can be shown that the class M_1 contains: Taylor sums; (C, m)-means for any natural m; B.-R.-means with multipliers $\mu_n(k) = \cos k\alpha_n$ $(0 \le k \le n)$, under the condition $\alpha_n = O(1/n)$; the Vallée-Poussin integral belongs to the class $M_1^{(1)}$, but does not belong to any of the classes $M_1^{(r)}$ for $r \ge 2$; Hölder means of order $m \ge 2$ do not belong to any of the classes $M_1^{(r)}$ $(r = 1, 2, \ldots)$.

The summation method $\mu \in M_1^{(r)}$ admits, for $A_n(z, f; \mu)$ $(f(z) \in B^{(r)})$, the representation

$$A_n(z, f; \mu) = \sum_{j=0}^{r-1} \alpha_j(n, z, f) + \frac{d_n}{n^r}\, p_{n-r}\big(z, f^{(r)}\big) + \frac{b_n}{n^r},$$

where d_n and b_n are uniformly bounded with respect to n, with respect to z from $|z| \le 1$, and with respect to the classes $B^{(r)}$ and $M_1^{(r)}$.

Now, for a given method $\mu \in M_1^{(r)}$ and $f(z) \in B^{(r)}$, construct the mean

$$B_n(z, f; \mu) = A_n(z, f; \mu) - \left[\sum_{j=0}^{r-1} \alpha_j(n, z, f) + \frac{d_n}{n^r}\, p_{n-r}\big(z, f^{(r)}\big)\right],$$

which, for $n \geq r-1$, gives for the function $f(z) \in B^{(r)}$ an approximation of the order of the best possible.

4. The theorem stated above and its corollaries are easily generalized to the classes of summation methods T_m $(m = 2, 3, ...)$. For Vallée-Poussin sums it makes it possible to obtain results analogous to those of S. M. Nikol' skii [4] and A. D. Shcherbina [5]. The method of investigation makes it possible to solve an analogous problem for other classes of functions.

In conclusion, the author expresses sincere gratitude to A. Kh. Turetskii, under whose supervision the present work was carried out. The author also thanks S. B. Stechkin for valuable advice and comments.

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Received
29 I 1957

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.