PHYSICS

G. A. SOKOLIK

1957

SovietRxiv

Abstract

Full Text

PHYSICS

G. A. SOKOLIK

ON THE QUESTION OF THE CONNECTION BETWEEN ORDINARY AND ISOTOPIC SPACES

(Presented by Academician N. N. Bogolyubov, February 13, 1957)

1. Let us develop a mathematical apparatus suitable for establishing a connection between ordinary and isotopic spaces. We proceed from the premise that isotopic spin and other invariants of the charge group may be interpreted as internal degrees of freedom of a particle. In this case the particle may be described by Pauli equations ([1]) or ([2]), as well as by more general equations ([3]), describing particles of arbitrary spatial and isotopic spins. One of the shortcomings of such equations is that they decompose into independent parts acting on the spatial and charge variables. It appears natural to develop a theory of equations generalizing the equations of works ([1–3]) and having the form:

$$\left(\Gamma_i \frac{\partial}{\partial x_i} + \chi\right)\psi = 0 \quad (i = 1, 2, \dots, n) \tag{1}$$

(x_1 will always be regarded as imaginary and identified with time). x_1, x_2, x_3, x_4 will be regarded as coordinates of Minkowski space. The remaining $n-4$ coordinates specify the isotopic space.

2. Let us solve the general problem of finding all equations (1) invariant with respect to rotations and reflections in n-dimensional space. The invariance conditions will take the form

$$[\Gamma_i I_{jk}] = g^{ij}\Gamma_k - g^{ik}\Gamma_j, \tag{2}$$

$$g^{nn} = -g^{11} = \dots = g^{n-1,n-1} = 1, \qquad g_{ik} = 0, \quad i \neq k,$$

where I_{ij} are infinitesimal operators of the representation of the n-dimensional orthogonal group found in ([4]). The problem we have posed of classifying equations according to representations of a continuous (in the present case orthogonal) group reduces ([5]) to finding the irreducible representations of the group into which the direct product of an arbitrary representation by the identical one (the n-dimensional vector) $\partial\psi_i/\partial x_j$ decomposes, where ψ_i transforms according to an arbitrary representation, and the gradient $\partial/\partial x_j$ according to the identical one.

3. According to ([4]), the basis vector of the representation is specified by arrays of numbers satisfying the inequalities:

$$\begin{gathered} m_{2p+1,i+1} \leq m_{2p,i} \leq m_{2p+1,i} \quad -m_{2p,p} \leq m_{2p-1,p} \leq m_{2p,p}, \\ m_{2p,i+1} \leq m_{2p-1,i} \leq m_{2p,i} \quad (i = 1, 2, \dots, p). \end{gathered} \tag{3}$$

Let us derive from this the dimensions of the representations.

For the case of a $(2k+2)$-dimensional group we have

$$S_{2k+2} = \prod_{1 \leq j \neq i \leq k+1} \frac{(p_i + p_j)(p_i - p_j)}{\alpha_k},$$

$$p_i = m_{2k+1,i} + k - i + 1 \quad (k = 1, 2, ...); \tag{4}$$

in the case of the $(2k+1)$-dimensional group

$$S_{2k+1} = \prod_{1 \leq j \neq i \leq k} (p_i - p_j)(p_i + p_j + 1) \prod_{i=1}^{k} \frac{2p_i + 1}{\beta_k} \quad (p_i = m_{2k,i} + k - i), \tag{4'}$$

where α_k and β_k have the form:

$$\alpha_k = \prod_{l=0}^{k-1} (k + l + 1)(k - l + 1) \prod_{j=1}^{k-1} \prod_{i=0}^{j-1} \frac{(j + i)(j - i)}{2k + 2}, \tag{5}$$

$$\beta_k = \prod_{i=0}^{k-2} (2i + 1) \prod_{l=0}^{k-2} (k + l + 1)(k - l) \prod_{j=1}^{k-2} \prod_{i=0}^{j-1} (j + i + 1)(j - i).$$

Using the fact that in the case of the $(2k+2)$-dimensional group the highest vector of the representation

$$\xi \begin{bmatrix} m_{2k+1,1} & \cdots & m_{2k+1,k} \\ \cdot & \cdot & \cdot \\ & m_{41} & m_{42} \\ & m_{31} & m_{32} \\ & & m_{21} \\ & & m_{11} \end{bmatrix}$$

is an eigenvector of the commuting operators of the representation $I_{2k+2,2k+1} \ldots I_{21}$, with the weights given by the relations

$$I_{2k+2,2k+2}\xi(\alpha) = m_{2k+1,k+1}\xi(\alpha) \ldots I_{21}\xi(\alpha) = m_{2k+1,1}\xi(\alpha), \tag{6}$$

we obtain the decomposition of the direct product of two representations of the $(2k+2)$-dimensional group into irreducibles. In the particular case when one of the representations τ_0 is the identity representation, we have:

$$[(m_{2k+1,1} \ldots m_{2k+1,k+1}) \times \tau_0] = (m_{2k+1,1} \pm 1 \ldots m_{2k+1,k+1}) + \ldots$$

$$\ldots + (m_{2k,1} \ldots m_{2k,k} \pm 1); \quad \tau_0 = (1, 0 \ldots 0). \tag{7}$$

For the $(2k+1)$-dimensional case we have analogously:

$$[(m_{2k,1} \ldots m_{2k,k}) \times \tau_0] = (m_{2k,1} \pm 1 \ldots m_{2k,k}) + \ldots$$

$$\ldots + (m_{2k,1} \ldots m_{2k,k} \pm 1) + \Delta_{m_{2k,k}}(m_{2k,1} \ldots m_{2k,k});$$

$$\Delta_{m_{2k,k}} = \begin{cases} 1, & m_{2k,k} \neq 0; \\ 0, & m_{2k,k} = 0. \end{cases}$$

We shall say that representations τ and τ' are *linked* if τ' is contained among the irreducible representations into which the product $[\tau \times (1, 0 \ldots 0)]$ decomposes. We thus obtain the following general rule: in the case of the $(2k+2)$-dimensional group a representation cannot be linked with itself; in the case of the $(2k+1)$-dimensional group this occurs if all $m_{2k,i}$ are different from zero.

4. To determine the matrices Γ_i in the $(2k+2)$-dimensional case it suffices to find the matrix Γ_{2k+2}. The remaining matrices are expressed through it by the formulas following from (2):

$$[\Gamma_{2k+2}, I_{2k+2,k+1}] = \Gamma_{2k+1} \ldots [\Gamma_{2k+2}, I_{2k+1,1}] = \Gamma_1.$$

The matrix Γ_{2k+2} has the form:

$$\Gamma_{2k+2}\xi(\alpha)_\tau = \sum_{\tau'\alpha'} C^{\tau\tau'}_{\alpha\alpha'}\xi(\alpha')_{\tau'}, \tag{8}$$

and satisfies the invariance condition following from (2):

$$[[\Gamma_{2k+2}, I_{2k+2,\,2k+1}]I_{2k+2,\,2k+1}] = \Gamma_{2k+2}. \tag{9}$$

Using the relations $0 = [\Gamma_{2k+2}, I_{2k+1,\,2k}] = \dots = [\Gamma_{2k+2}, I_{21}]$, also following from (2), we write (8), according to Schur' s lemma, in the form

$$\Gamma_{2k+2}\xi(\alpha)_\tau = \sum_{\tau'} C_{\tau\tau'}^{m_{2k,1}\cdots m_{2k,k}} \xi(\alpha)_{\tau'}. \tag{10}$$

Substituting formula (10) and the matrix coefficients of the infinitesimal representation from (4) into (9), we obtain a system of homogeneous equations for $C_{\tau\tau'}^{m_{2k,1}\cdots m_{2k,k}}$. The system consists of $k!/(k-2)!2!$ subsystems, each of which corresponds to a pair of numbers $m_{2k,i}$ and $m_{2k,j}$ and is formed with respect to the different

$$C_{\tau\tau'}^{m_{2k,i}\pm1, m_{2k,j}}, \quad C_{\tau\tau'}^{m_{2k,i} m_{2k,j}\pm1}, \quad C_{\tau\tau'}^{m_{2k,i}\pm1, m_{2k,j}\pm1}, \quad C_{\tau\tau'}^{m_{2k,i} m_{2k,j}}.$$

As was shown above, the solutions of the system can be different from zero only in the case when τ and τ' are linked representations, since if an irreducible representation τ is contained in the representation according to which ψ is transformed, then τ' must enter the representation according to which $\partial\psi/\partial x_j$ is transformed.

Substituting the linkage conditions (7), we have, for the $(2k+2)$-dimensional space, the general formula

$$C_{\tau\tau'}^{m_{2k,1}\cdots m_{2k,k}} = C_{\tau\tau'} \times \tag{11}$$

$$\times \prod_{1\le j<k} \sqrt{(m_{2k,j} + m_{2k+1,j} + 2k - i - j + 2)(m_{2k,j} - m_{2k+1,i} + i - j - 1)},$$

$$m'_{2k+1,i} = m_{2k+1,i} + 1; \qquad m'_{2k+1,j} = m_{2k+1,j} \quad (i \ne j),$$

where $C_{\tau\tau'}$ are arbitrary complex constants.

In the case of the $(2k+1)$-dimensional group, arguing analogously, one can obtain the general formula for the matrix coefficients Γ_{2k+1}:

$$C_{\nu\nu'}^{m_{2k-1,1}\cdots m_{2k-1,k}} = C_{\nu\nu'} \times \tag{12}$$

$$\prod_{1\le j<k} \times \sqrt{(m_{2k,i} + m_{1k-1,i} + 2k - j - i + 1)(m_{2k,i} - m_{2k-1,j} - j + i - 1)},$$

if

$$m'_{2k,i} = m_{2k,i} + 1; \qquad m'_{2k,j} = m_{2k,j}, \quad i \neq j;$$

$$C^{m_{2k-1,1} \cdots m_{2k-1,k}}_{\nu\nu'} = C_{\nu\nu'} \prod_{1 \le j < k} m_{2k-1}, \tag{12'}$$

if

$$m'_{2k,i} = m_{2k,i},$$

where ν are irreducible representations of the $(2k+1)$-dimensional group.*

5. In conclusion, let us find the conditions of invariance of equation (1) with respect to the inversions $x' = -x,\ x'_n = x_n$ or $x' = x,\ x'_n = -x_n$, where x denotes the remaining $n-1$ coordinates. The operators T, for $2k+2 = n$, satisfy the relations

$$[TI_{2k+2,\,2k+1}]_+ = 0; \qquad [TI_{2k+1,\,2k}] = \cdots = [TI_{2,1}] = 0. \tag{13}$$

It can be shown that in the case of the $(2k+2)$-dimensional group $\tau = (m_{2k+1,1} \cdots m_{2k+1,k+1})$ under inversion passes into $\bar{\tau}(m_{2k+1,1} \cdots - m_{2k+1,k+1})$. If

* The computed formulas give the Clebsch–Gordan coefficients describing the decompositions of the product of an arbitrary representation of the n-dimensional orthogonal group by an n-dimensional vector and coinciding with Γ_i.

$m_{2k+1,k+1} = 0$, τ and $\dot{\tau}$ coincide. In the first case the operator T has the form

$$T \begin{pmatrix} 0 & (-1)^{\sum \lambda_i} \\ (-1)^{\sum \lambda_i} & 0 \end{pmatrix}, \qquad \lambda_i = m_{2k,i} + \begin{cases} 1 \\ 0 \end{cases} \qquad (i = 1, 2, \dots, k). \tag{14}$$

The matrix acts on the vector

$$\begin{pmatrix} \xi^{(\alpha)\tau} \\ \xi^{(\alpha)\dot{\tau}} \end{pmatrix}.$$

In the second case we have $T = (-1)^{\sum \lambda_i}$. Expanding the condition of invariance under inversions $[T\Gamma_{2k+2}] = 0$, we obtain conditions on $C_{\tau\tau'}$ coinciding with the invariance conditions in the case $n = 4$, given in (6). In the case of a $(2k+1)$-dimensional group, τ and $\dot{\tau}$ coincide, and

$$T = (-1)^{\sum \lambda_i}\, {}^*.$$

6. Thus, the matrices Γ_{2k+2} and Γ_{2k+1} consist of boxes numbered by irreducible representations of $(2k+2)$- and $(2k+1)$-dimensional groups, specified by the numbers $m_{2k+1,1} \dots m_{2k+1,k+1}$ and $m_{2k,1} \dots m_{2k,k}$, respectively:

$$\left\| C_{\tau\tau'}^{m_{2k,1} \cdots m_{2k,k}} \right\| \times I_{2k+1}; \qquad \left\| C_{\nu\nu'}^{m_{2k-1,1} \cdots m_{2k-1,k}} \right\| \times I_{2k}. \tag{15}$$

I_{2k+1} and I_{2k} are identity matrices of dimension S_{2k+1} or S_{2k} (4), (4′). In the case when Γ_i is reducible to diagonal form with eigenvalues different from zero, and, as is easily seen from (15), each eigenvalue has multiplicity S_{2k+1} or S_{2k}, this gives us the right to interpret $m_{2k,1} \dots m_{2k,k}$ or $m_{2k-1,1} \dots m_{2k-1,k}$ as a generalization of spatial spin. The numbers may be identified with the values of spin, isotopic spin, and other invariants of an isotopic group of dimension higher than three. In the case of an 8-dimensional group, $C_{\tau\tau'}^{m_{61} m_{62} m_{63}}$ will be specified by three numbers, which may be associated with spin, isotopic spin, and Gell-Mann strangeness (the isotopic space in this case is 4-dimensional). As an application of the unified description of ordinary and isotopic spaces, one may point to the possibility of interpreting the well-known paradox connected with nonconservation of parity of K mesons (8). Using (14) and changing the value λ_2 by one (the case $n = 8$ is considered), one can compensate the change of parity in Minkowski space by a change of parity in isotopic space. Thus, from this point of view, the paradox is explained by the fact that the parity of K mesons refers to inversion in a space uniting Minkowski space and isotopic space.

In conclusion I express my gratitude to Academician N. N. Bogolyubov, Prof. D. D. Ivanenko, and M. L. Tsetlin for their interest in the work and for fruitful discussions.

Moscow State University
named after M. V. Lomonosov

Received
12 II 1957

REFERENCES

1. A. Pais, *Physica*, **19**, 8, 69 (1953).

2. D. Ivanenko, G. Sokolik, DAN, **97**, No. 4 (1954).

3. I. E. Tamm, V. L. Ginzburg, ZhETF, **17**, 227 (1947).

4. I. M. Gelfand, M. L. Tsetlin, DAN, **71**, No. 6 (1950).

5. É. Cartan, *Theory of Spinors*, IL, 1947.

6. B. L. Van-der-Waerden, *Die gruppentheoretische Methode in der Quantenmechanik*, Berlin, 1932.

7. I. M. Gelfand, A. M. Yaglom, ZhETF, **18**, 703 (1948).

8. T. D. Lee, C. N. Yang, *Phys. Rev.*, **102**, No. 1, 290 (1956).

* The ordinary parity operator corresponding to inversion in space-time, T_0, commutes with $I_{2k+2,2k+1}$.

Note: Figure translations are in progress. See original paper for figures.