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MATHEMATICS

M. A. EVGRAFOV and A. D. SOLOV' EV

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Abstract

Full Text

MATHEMATICS

M. A. EVGRAFOV and A. D. SOLOV' EV

ON A GENERAL CRITERION FOR A BASIS

(Presented by Academician A. N. Kolmogorov on 12 X 1956)

We shall say that a system of functions

$$u_n(z) = z^n \varphi_n(z), \quad \varphi_n(0) = 1, \quad n = 0, 1, 2, \dots, \quad (1)$$

regular in a domain G , forms a basis in a domain $G_1 \subset G$, if every function $f(z)$, regular in G_1 , is represented in this domain by a uniformly convergent series

$$f(z) = \sum_{n=0}^{\infty} a_n u_n(z),$$

and this representation is unique.

Theorem 1. *Let a system (1) be given, where the functions $\varphi_n(z)$ are regular in the disk $|z| < R$ and do not vanish inside this disk.*

Write the system (1) in the form

$$u_n(z) = z^n e^{-\lambda_n(z)},$$

where the functions $\lambda_n(z)$ are regular in the disk $|z| < R$.

Introduce the notation

$$\lambda_n(z) - \lambda_{n-1}(z) = \Delta_n(z) = \sum_{k=1}^{\infty} \Delta_{nk} z^k, \quad \Delta_0(z) = \lambda_0(z);$$

$$\Delta_n^0(r) = \sum_{k=1}^{\infty} |\Delta_{nk}| r^k, \quad l_n(r) = \sum_{k=0}^n \Delta_k^0(r). \quad (2)$$

If the functions $\lambda_n(z)$ satisfy the condition

$$\lim_{n \rightarrow \infty} \frac{l_n(r)}{n} = 0 \quad \text{for every } r < R, \quad (3)$$

then the system (1) forms a basis in the disk $|z| < R$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an arbitrary function regular in the disk $|z| < R$. Write the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n e^{-\lambda_n(z)}.$$

From this formal identity, by comparing coefficients of like powers of z , one can obtain finite recurrence relations from which the numbers a_n are determined successively (and moreover uniquely).

By virtue of condition (3),

$$\lim_{n \rightarrow \infty} \sqrt[n]{|z^n \varphi_n(z)|} = |z|, \quad |z| < R,$$

therefore, in order to prove the theorem it is enough to show that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \leq \frac{1}{R}.$$

Introduce the numbers a_{nk} as the coefficients of the series

$$f(z) = \sum_{n=0}^{k-1} a_{nk} z^n e^{-\lambda_n(z)} + \sum_{n=k}^{\infty} a_{nk} z^n e^{-\lambda_k(z)}.$$

It is not difficult to verify that the property of the system (1) of being a basis in the disk $|z| < R$ will not be violated if we change a finite number of functions $\varphi_n(z)$ (while, of course, preserving the conditions: $\varphi_n(z)$ are regular for $|z| < R$ and $\varphi_n(0) = 1$). Therefore our series converges in the same disk in which the series

$$\sum_{n=0}^{\infty} \tilde{a}_{nk} z^n = f(z) e^{\lambda_k(z)},$$

does, i.e. for $|z| < R$.

We note that $a_{nk} = a_n$ for $n < k + 1$, since the coefficients a_n depend only on the first n functions $\lambda_0(z), \lambda_1(z), \dots, \lambda_{n-1}(z)$.

Write two identities:

$$f(z) = \sum_{n=0}^{k-1} a_n z^n e^{-\lambda_n(z)} + \left(\sum_{n=k}^{\infty} a_{nk} z^n \right) e^{-\lambda_k(z)},$$

$$f(z) = \sum_{n=0}^{k-1} a_n z^n e^{-\lambda_n(z)} + \left(\sum_{n=k}^{\infty} a_{n,k-1} z^n \right) e^{-\lambda_{k-1}(z)}.$$

Comparing them, we obtain

$$\sum_{n=k}^{\infty} a_{nk} z^n = \left(\sum_{n=k}^{\infty} a_{n,k-1} z^n \right) e^{\Delta_k(z)}. \quad (4)$$

We shall prove that every $a_{nk} = a_{nk}(\Delta_{ij}; c_m)$ is a polynomial in the Δ_{ij} and c_m with positive coefficients. We carry out the proof by induction on the index k .

$$\sum_{n=0}^{\infty} a_{n0} z^n = f(z) e^{\Delta_0(z)},$$

so that for the numbers a_{n0} this assertion is true. Suppose it is true for the numbers $a_{n,k-1}$. Then from identity (4) it follows that it is true also for the numbers a_{nk} .

Now assume that all coefficients c_m and Δ_{ij} are nonnegative. Then all $a_{nk} \geq 0$.

Denote

$$f_k(z) = \sum_{n=k}^{\infty} a_{nk} z^n.$$

From (4) it follows that

$$f_k(z) \ll f_{k-1}(z) e^{\Delta_k(z)}, \quad (5)$$

where the symbol \ll means that the series on the left is majorized by the series on the right, i.e. the coefficients of the left-hand series do not exceed the corresponding coefficients of the right-hand series.

From (5) it follows immediately that

$$f_k(z) \ll e^{\lambda_k(z)} f(z),$$

whence we obtain

$$a_{nk} \ll A_{nk} = \frac{1}{2\pi i} \int_{|z|=r < R} \frac{f(z) e^{\lambda_k(z)}}{z^{n+1}} dz \ll \frac{e^{\lambda_k(r)}}{r^n} f(r).$$

If now c_m and Δ_{ij} are arbitrary complex numbers, then

$$|a_{nk}(\Delta_{ij}, c_m)| \ll a_{nk}(|\Delta_{ij}|; |c_m|) \ll \frac{e^{l_k(r)}}{r^n} M_0(r),$$

where

$$M_0(r) = \sum_{m=0}^{\infty} |c_m| r^m.$$

In particular,

$$|a_n| \ll M_0(r) \frac{e^{l_n(r)}}{r^n}, \quad r < R. \quad (6)$$

From this inequality it follows that

$$\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|} \ll \frac{1}{r},$$

whence, by virtue of the arbitrariness of $r < R$,

$$\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|} \ll \frac{1}{R}.$$

The theorem is proved.

Remark 1. Theorem 1 will also be valid in the case when, in each disk $|z| \leq r < R$, only a finite number of the functions $\varphi_n(z)$ have zeros, since we can always change the functions having zeros for $|z| \leq r$ (putting, for example, $\varphi_n(z) = 1$) without violating the property of system (1) of being a basis in the disk $|z| < R$.

Remark 2. It can be shown that if $|z| < R$ is the largest disk in which the conditions of Theorem 1 are fulfilled (weakened by Remark 1), then for the given functions $\Delta_n^0(r)$ there exists a system (1) which in the disk $|z| < R + \varepsilon$, for any $\varepsilon > 0$, no longer forms a basis.

Remark 3. Inequality (6) does not depend on condition (3) and has a universal character.

From Theorem 1 the following follows without difficulty:

Theorem 2. Suppose a system is given

$$u_n(z) = u^n(z) \psi_n(z), \quad \psi_n(0) = 1, \quad n = 0, 1, \dots, \quad (7)$$

where $u(z) = z + \dots$ is a function regular and univalent in a simply connected domain G , which is mapped by the function $u(\zeta)$ onto a disk with center at the

origin. The functions $\psi_n(z)$ are regular in the domain G , and on each closed set $E \subset G$ only a finite number of these functions have zeros. Denote

$$l_n(E) = \sum_{k=k_0}^n \max_{z \in E} \left| \ln \left[\frac{\psi_{k-1}(z)}{\psi_k(z)} \right] \right|$$

($E \subset G$ is a closed set, $k_0 = k_0(E)$).

If

$$\lim_{n \rightarrow \infty} \frac{l_n(E)}{n} = 0 \quad \text{for every } E \subset G,$$

then the system (7) forms a basis in the domain G .

Theorem 3. If in Theorem 1 condition (3) is replaced by the condition

$$\overline{\lim}_{n \rightarrow \infty} \frac{l_n(r)}{n} = l(r),$$

then the system (1) will form a basis in the disk

$$|z| e^{l_1(|z|)} < \rho,$$

where

$$\rho = \sup_{r < R} r e^{-l(r)}, \quad l_1(r) = \overline{\lim}_{n \rightarrow \infty} \frac{\max_{|z|=r} |\lambda_n(z)|}{n} \ll l(r).$$

Let us consider several examples of applications of these theorems.

1.

$$u_n(z) = z^n \varphi(z) e^{\lambda_n z}. \quad (8)$$

Suppose that

$$\sum_{k=0}^n |\lambda_k - \lambda_{k-1}| = o(n).$$

In this case the system forms a basis in the disk $|z| < |\alpha_1|$, where α_1 is the zero of the function $\varphi(z)$ nearest to the origin.

2.

$$u_n(z) = u^n(z)e^{\lambda_n z}, \quad u(z) = z + az^2 + \dots. \quad (9)$$

Again require that

$$\sum_{k=0}^n |\lambda_k - \lambda_{k-1}| = o(n).$$

Then the system (9) forms a basis in the domain G in which the function $\mu(z)$ is univalent and which is mapped by the function $u(z)$ onto a disk with center at the origin.

3.

$$u_n(z) = z^n \varphi^{\lambda_n}(z), \quad \varphi(0) = 1. \quad (10)$$

Let, again,

$$\sum_{k=0}^n |\lambda_k - \lambda_{k-1}| = o(n).$$

Then the system will form a basis in the disk $|z| < |\alpha_1|$ (α_1 is the zero of the function $\varphi(z)$ nearest to the origin).

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