



Soviet-era science, translated into English

MATHEMATICS

Ya. S. BUGROV

1957

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-195701.20000>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

Ya. S. BUGROV

THE DIRICHLET PROBLEM FOR A DISK

(Presented by Academician A. N. Kolmogorov on 14 III 1957)

1. The present paper is devoted to clarifying the differential properties of polyharmonic functions in a disk as dependent on the differential properties of the boundary functions.

We consider the Dirichlet problem for the equation

$$\Delta^l U(\rho, \theta) = 0 \quad (l = 1, 2, \dots) \quad \{0 \leq \rho \leq 1; 0 \leq \theta \leq 2\pi\}; \quad (1)$$

$$\left. \frac{\partial^k U(\rho, \theta)}{\partial \rho^k} \right|_{\rho=1} = \varphi_k(\theta) \quad (k = 0, 1, \dots, l-1), \quad (2)$$

where Δ is the Laplace operator.

We have proved the following theorem, which generalizes the corresponding results of S. M. Nikol'skii⁽¹⁾ and T. I. Amanov⁽²⁾.

Theorem 1. If the 2π -periodic functions $\varphi_k(\theta) \in H_p^{r+l-(k+1)}(M)$ ($k = 0, 1, \dots, l-1$), then the polyharmonic function

$$U(\rho, \theta) \in H_p^{r+1/p+l-1}(cM),$$

where $r > 0$, $1 \leq p \leq \infty$ (for $l > 1$, r is not an integer)*.

For $p = 2$ and $l = 1$ this is the result of S. M. Nikol'skii⁽¹⁾. For $p = 2$ and $l = 2$ our result somewhat strengthens the corresponding result of T. I. Amanov⁽²⁾.

We note that our result for $l = r + 1/2$ ($p = 2$) overlaps with the corresponding result of V. M. Babich and L. N. Slobodetskii⁽⁴⁾. It does not follow from the result of these authors, just as their result does not follow exactly from ours. We also note the investigations related to these questions by N. I. Mozhherova⁽⁵⁾ and O. V. Besov⁽⁶⁾.

2. Consider a harmonic polynomial of order n in the unit disk

$$\Phi_n(\rho, \theta) = \sum_0^n \rho^k (a_k \cos k\theta + b_k \sin k\theta). \quad (3)$$

Denote

$$\|\Phi\|_{L_p(R)} = \left(\int_R^1 \int_0^{2\pi} |\Phi(\rho, \theta)|^p \rho \, d\rho \, d\theta \right)^{1/p} \quad (R \geq 0); \quad \|\Phi_n\|_p = \|\Phi_n\|_{L_p(0)}.$$

Lemma 1. The inequalities

$$\left\| \frac{\partial^l \Phi_n}{\partial \rho^l} \right\|_{L_p(R)} \leq C_R n^l \|\Phi_n\|_{L_p(R)}, \quad (4)$$

hold

* For the definition of the classes $H_p^r(M)$, see (3).

where C_R is a constant depending on R and l ;

$$\left\| \rho^l \frac{\partial^l \Phi_n(\rho, \theta)}{\partial \rho^l} \right\|_p \leq cn^l \|\Phi_n\|_p; \quad (5)$$

$$\left\| \frac{\partial^l \Phi_n}{\partial \theta^l} \right\|_p \leq n^l \|\Phi_n\|_p, \quad (6)$$

where $l = 1, 2, \dots; 1 \leq p \leq \infty$.

Proof. From (3) we have

$$\frac{\partial^l \Phi_n}{\partial \rho^l} = \sum_{i=1}^l \Psi_n^{(i)}(\rho, \theta),$$

where

$$\Psi_n^{(i)}(\rho, \theta) = \frac{\lambda_i}{\rho^l} \sum_{k=1}^n k^i \rho^k (a_k \cos k\theta + b_k \sin k\theta)$$

and λ_i ($i = 1, \dots, l$) are the corresponding constants. Obviously,

$$\begin{aligned} \Psi_n^{(i)}(\rho, \theta) &= \frac{\lambda_i}{\pi \rho^l} \int_0^{2\pi} \Phi_n(\rho, u + \theta) \sum_1^n k^i \cos ku \, du = \\ &= \frac{2\lambda_i n^i}{\pi \rho^l} \int_0^{2\pi} \cos nu \cdot \Phi_n(\rho, u + \theta) \sigma_n^{(i)}(u) \, du, \end{aligned} \quad (7)$$

where

$$\sigma_n^{(i)}(u) = \sum_0^{n-1} \left(1 - \frac{k}{n}\right)^i \cos ku$$

and the prime means that for $k = 0$ the corresponding term of the sum is equal to $1/2$.

Applying Abel's transformation to the last sum, we obtain

$$\begin{aligned} \sigma_n^{(i)}(u) &= \sum_{k=0}^{n-3} (k+1) \Delta^2 \left(1 - \frac{k}{n}\right)^i F_k(u) + \\ &+ \frac{1}{n^{i-1}} F_{n-1}(u) + \frac{n-1}{n^i} (2^i - 2) F_{n-2}(u), \end{aligned}$$

where $F_k(u)$ are the Fejér kernels, and $\Delta^2 \mu_k = \mu_k - 2\mu_{k+1} + \mu_{k+2}$.

It is important to note that $F_k(u) \geq 0$ and $\Delta^2 \left(1 - \frac{k}{n}\right)^i \geq 0$, whence $\sigma_n^{(i)}(u) \geq 0$ and

$$\frac{1}{\pi} \int_0^{2\pi} |\sigma_n^{(i)}(u)| \, du = 1 \quad (i = 1, 2, \dots).$$

Applying to (7) the generalized Minkowski inequality (7; p. 246), we obtain

$$\|\Psi_n^{(i)}\|_{L_p(R)} \leq 2|\lambda_i| n^i R^{-l} \|\Phi_n\|_{L_p(R)} \frac{1}{\pi} \int_0^{2\pi} |\sigma_n^{(i)}(u)| \, du \leq C_R n^l \|\Phi_n\|_{L_p(R)},$$

whence (4) and (5) follow. As for inequality (6), it follows from the fact that $\Phi_n(\rho, \theta)$ is a trigonometric polynomial in θ (1³). For $p = \infty$, inequalities (4), (5), and (6), which are analogues of Bernstein's inequality, were proved by A. L. Shaginyan (8).

3. Proof of Theorem 1 for $l = 1$. In this case the solution of the boundary-value problem (1), (2) can be written in the form ($\varphi = \varphi_0$)

$$U(\rho, \theta) = \frac{1}{2\pi} \int_0^{2\pi} K(\rho, u) \varphi(u + \theta) du,$$

where

$$K(\rho, \theta) = \sum_0^{\infty} \rho^k \cos k\theta = \sum_0^{\infty} (k+1) \Delta_{\rho}^2 \rho^k F_k(\theta) \quad (0 \leq \rho < 1).$$

Let

$$\varphi(\theta) \sim \sum_0^{\infty} A_k(\theta) = \sum_0^{\infty} (a_k \cos k\theta + b_k \sin k\theta),$$

$$S_n = S_n(\varphi; \theta) = \sum_0^n A_k(\theta), \quad \tau_n(\varphi; \theta) = \frac{S_n + \dots + S_{2n-1}}{n}.$$

Then

$$\begin{aligned} \|\tau_n(\varphi; \theta)\|_p^* &= \left(\frac{1}{2\pi} \int_0^{2\pi} |\tau_n(\varphi; \theta)|^p d\theta \right)^{1/p} = \\ &= \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{\pi} \int_0^{2\pi} [2F_{2n-1}(u) - F_{n-1}(u)] \varphi(u + \theta) du \right|^p d\theta \right)^{1/p} \leq B \|\varphi\|_p^*, \end{aligned} \quad (8)$$

where B is a constant. This follows, after applying the generalized Minkowski inequality, from the fact that the Fejér kernel is bounded in the L metric.

Further, since $\varphi \in H_p^r(M)$, there exists a trigonometric polynomial $T_n(\theta)$ of order n such that

$$\|\varphi - T_n\|_p^* \leq \frac{cM}{n^r}. \quad (9)$$

From (8), (9), and the fact that $\tau_n(T_n; \theta) = T_n(\theta)$ for all trigonometric polynomials of order n , it follows that

$$\|\varphi(\theta) - \tau_n(\varphi; \theta)\|_p^* \leq \frac{cM(B+1)}{n^r}. \quad (10)$$

Construct the harmonic polynomial of order $(2n - 1)$

$$\Phi_n(\rho, \theta) = \frac{1}{2\pi} \int_0^{2\pi} K(\rho, u) \tau_n(\varphi; u + \theta) du.$$

Then

$$\begin{aligned} U(\rho, \theta) - \Phi_n(\rho, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} K(\rho, u) [\varphi(\theta + u) - \tau_n(\varphi; u + \theta)] du = \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n+1}^{\infty} (k+1) \Delta_\rho^2 \rho^k F_k(u) [\varphi(u + \theta) - \tau_n(\varphi; u + \theta)] du. \end{aligned}$$

Hence, applying the generalized Minkowski inequality, we obtain

$$\begin{aligned} \|U - \Phi_n\|_p &\leq \|\varphi - \tau_n\|_p^* \left(\int_0^1 \left| \sum_{n+1}^{\infty} (k+1) \Delta_\rho^2 \rho^k \right|^p d\rho \right)^{1/p} = \\ &= \|\varphi - \tau_n\|_p^* \left(\int_0^1 |(n+1) \Delta \rho^{n+1} + \rho^{n+1}|^p d\rho \right)^{1/p} \leq \\ &\leq \frac{cM(B+1)}{n^r} \left[(n+1) \left(\frac{\Gamma(p+1)\Gamma(np+p+1)}{\Gamma(np+2p+2)} \right)^{1/p} + \left(\frac{1}{(n+1)p} \right)^{1/p} \right], \end{aligned}$$

where $\Gamma(u)$ is the gamma function; $\Delta_\rho^k = \rho^k - \rho^{k+1}$.

Consequently,

$$\|U - \Phi_n\|_p \leq \frac{c_1 M}{n^{r+1/p}} \quad (1 \leq p \leq \infty). \quad (11)$$

By the well-known Bernstein method, by means of which the converse theorem on best approximation is proved, it follows easily from (11), using Lemma 1, that $U(\rho, \theta)$ belongs to the class $H_p^{r+1/p}(c_1 M)$ of functions defined on the annulus $1/2 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$, and consequently also to the class of functions defined in the disk $0 \leq \rho \leq 1$, since $U(\rho, \theta)$ is harmonic for $\rho < 1$ (see, for example, (7), p. 264).

4. Let $l > 1$. It is known that a polyharmonic function in the disk can always be represented in the form

$$U(\rho, \theta) = \sum_{k=0}^{l-1} (1 - \rho^2)^k U_k(\rho, \theta),$$

where the $U_k(\rho, \theta)$ are harmonic functions.

Since the arguments are the same for any l , for simplicity we consider the case $l = 2$. The solution of the boundary-value problem (1), (2) for $l = 2$ has the form

$$U(\rho, \theta) = (1 - \rho^2)U_1(\rho, \theta) + U_0(\rho, \theta). \quad (12)$$

The harmonic functions U_0, U_1 are determined from the following conditions:

$$U_0|_{\rho=1} = \varphi_0(\theta), \quad U_1|_{\rho=1} = \frac{1}{2}\varphi_1(\theta) - \left[\rho \frac{\partial U_0}{\partial \rho} \right]_{\rho=1}. \quad (13)$$

By the result of item 3, $U_0 \in H_p^{r+1+1/p}(cM)$.

By the embedding theorem of S. M. Nikol'skii (3),

$$\left. \frac{\partial U_0}{\partial \rho} \right|_{\rho=1} \in H_p^r(cM),$$

and $\varphi_1(\theta) \in H_p^r(M_1)$ by hypothesis; consequently,

$$U_1(\rho, \theta) \in H_p^{r+1/p}(c(M + M_1)).$$

By virtue of (12), the biharmonic function $U(\rho, \theta) \in H_p^{r+1/p+1}(c(M + M_1))$.

Thus Theorem 1 is completely proved.

In conclusion I express my sincere gratitude to my teacher S. M. Nikol'skii for his help in the work.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
7 XII 1957

REFERENCES

1. S. M. Nikol'skii, *Matem. sborn.*, **40** (82), no. 3, 303 (1956); **43** (85), no. 1 (1957).
2. T. I. Amanov, *DAN*, **88**, no. 3, 389 (1953).
3. S. M. Nikol'skii, *Matem. sborn.*, **33** (75), 261 (1953).
4. V. M. Babich, L. N. Slobodetskii, *DAN*, **106**, no. 4, 604 (1956).
5. N. I. Mozzherova, *Boundary properties of harmonic functions in three-dimensional space*, Dissertation, Mathematical Institute named after V. A. Steklov, Academy of Sciences of the USSR, 1956.
6. O. V. Besov, *Izv. AN SSSR, ser. matem.*, **20**, 469 (1956).
7. S. M. Nikol'skii, *Tr. Matem. inst. named after V. A. Steklov, AN SSSR*, **38**, 244 (1951).
8. A. L. Shaginyan, *DAN*, **90**, 141 (1953).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.