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Abstract

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ON NEW SUFFICIENT CONDITIONS FOR STABILITY, SEMISTABILITY, AND INSTABILITY OF A LIMIT CYCLE OF THE EQUATION

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$$

(Presented by Academician P. S. Aleksandrov on 27 IV 1957)

In the present work analytic criteria are obtained for the stability, semistability, and instability of a limit cycle of the equation

$$\frac{dy}{dx} = f(x,y), \tag{1}$$

where $f(x,y) = P(x,y)/Q(x,y)$, under the assumption that $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives of the required order.

In the paper of P. N. Papusha ([1]) the first step was made in the study of the question of semistability of cycles: a necessary condition for semistability of a regular cycle was found. V. A. Chechik ([2]) obtained a certain sufficient condition for semistability. Studying these works, the author observed that their method can be applied to the investigation not only of semistability, but also of stability and instability. As special cases of the main result of the present work one can obtain the criteria of the cited works.

1. Let L be a closed integral curve of equation (1):

$$x = \varphi(s), \qquad y = \psi(s) \qquad (0 \leq s \leq l),$$

its parametric representation (s is the arc length of the closed curve L, measured from some initial point counterclockwise).

Then, introducing a new system of coordinates in a sufficiently small neighborhood of L in such a way that each point of the neighborhood is uniquely determined by a segment n of the directed normal passing through this point, and by the arc length s of our curve, measured from the initial point to the foot of the indicated normal, equation (1) can be written ([1]) in the new coordinates as

$$\frac{dn}{ds} = F(s, n), \tag{2}$$

where

$$F(s, n) = \frac{\psi'(s) - n\varphi''(s) - f\varphi'(s) - nf\psi''(s)}{\varphi' + f\psi'}.$$

Following V. A. Chechik ([2]), write the increment of the coordinate n of the point $(0, n_0)$ of an integral curve after one circuit in the form

$$\psi(n_0) = \int_0^l F(s, n(s, n_0))\, ds.$$

Then ψ is a function of the point $n(0) = n_0$.

Since $\psi(0) = 0$, from the behavior of the function $\psi(n_0)$ for $n_0 > 0$ and $n_0 < 0$ close to zero one can judge the stability, semistability, or instability of the limit cycle L.

Obviously, for stability (instability) of the cycle it is sufficient that the condition

$$\psi'_{n_0}(n_0)\big|_{n_0=0} = 0, \qquad \psi''_{n_0}(n_0)\big|_{n_0=0} = 0, \dots, \qquad \psi^{(k-1)}_{n_0}(n_0)\big|_{n_0=0} = 0, \tag{3}$$

$$\psi^{(k)}_{n_0}(n_0)\big|_{n_0=0} < 0\ (> 0),$$

be fulfilled, where k is some odd number; and for semistability, the conditions

$$\psi'_{n_0}(n_0)\big|_{n_0=0} = 0, \qquad \psi''_{n_0}(n_0)\big|_{n_0=0} = 0, \tag{4}$$

$$\dots, \psi^{(k-1)}_{n_0}(n_0)\big|_{n_0=0} = 0, \qquad \psi^{(k)}_{n_0}(n_0)\big|_{n_0=0} \neq 0,$$

where k is even.

2. We shall seek a convenient expression for our derivatives.

$$\psi'_{n_0}(n_0) = \int_0^l F'_{n_0}(s, n(s, n_0))\, ds = \int_0^l F'_n(s, n) n'_{n_0}(s, n_0)\, ds.$$

Taking (2) into account, we easily obtain

$$n'_{n_0}(s) = 1 + \int_0^s F'_n(s, n) n'_{n_0}(s)\, ds.$$

Considering this relation as an integral equation with respect to $n'_{n_0}(s, n_0)$, we find its solution in the form

$$n'_{n_0}(s, n_0) = \exp\left[\int_0^s F'_n(t, n)\, dt\right].$$

Consequently,

$$\psi'_{n_0}(n_0) = \exp\left[\int_0^s F'_n(t, n)\, dt\right]_0^l.$$

Thus, the first sufficient condition for stability (instability) of the cycle L will be the condition

$$\int_0^l F'_n(s, n)\, ds\bigg|_{n=0} < 0 \ (> 0),$$

for at $n_0 = 0$ the traversal along the curve L is performed ($n(s, 0) \equiv 0$).

The main result of the work ([2]) is the finding of the first sufficient condition for semistability of a limit cycle of equation (1). It can be obtained from our conditions (4) for $k = 2$. The second derivative has the form

$$\psi''_{n_0}(n_0)\big|_{n_0=0} = \int_0^l F''_n \exp\left[\int_0^s F'_n\, dt\right] ds\bigg|_{n=0},$$

therefore the cycle will be semistable when

$$\int_0^l F'_n(s, n)\, ds\bigg|_{n=0} = 0, \qquad \int_0^l F''_n \exp\left[\int_0^s F'_n\, dt\right] ds\bigg|_{n=0} \neq 0.$$

It turns out that the higher derivatives can also be obtained in an equally simple form. Namely,

$$\psi_{n_0}^{(k)}(n_0)\big|_{n=0} = \int_0^l F_n^{(k)} \left[\exp \int_0^s F_n'\, dt\right]^{k-1} ds.$$

From the formula obtained and the conditions (3), (4), the following theorem follows.

Theorem 1. For stability (instability) of the cycle it is sufficient that the conditions

$$\int_0^l F_n'(s,n)\, ds\Big|_{n=0} = 0, \qquad \int_0^l F_n'' \exp\left[\int_0^s F_n'\, dt\right] ds\Big|_{n=0} = 0, \ldots$$

$$\ldots, \int_0^l F_n^{(k-1)} \left[\exp\left[\int_0^s F_n'\, dt\right]\right]^{k-2} ds\Big|_{n=0} = 0, \quad \int_0^l F_n^{(k)} \left[\exp\left[\int_0^s F_n'\, dt\right]\right]^{k-1} ds\Big|_{n=0} < 0\ (> 0) \tag{5}$$

hold for odd k, and for semistability—the conditions

$$\int_0^l F_n'(s,n)\, ds\Big|_{n=0} = 0, \qquad \int_0^l F_n'' \exp\left[\int_0^s F_n'\, dt\right] ds\Big|_{n=0} = 0, \ldots$$

$$\ldots, \int_0^l F_n^{(k-1)} \left[\exp\left[\int_0^s F_n'\, dt\right]\right]^{k-2} ds\Big|_{n=0} = 0, \quad \int_0^l F_n^{(k)} \left[\exp\left[\int_0^s F_n'\, dt\right]\right]^{k-1} ds\Big|_{n=0} \neq 0 \tag{6}$$

hold for even k.

Remark. If $\psi_{n_0}^{(k)}(n_0)\big|_{n_0=0}$ determines one or another type of stability of the cycle L, then the equalities

$$\int_0^l F_n'(s,n)\, ds\Big|_{n=0} = 0, \qquad \int_0^l F_n''(s,n) \exp\left[\int_0^s F_n'\, dt\right] ds\Big|_{n=0} = 0, \ldots$$

$$\ldots, \int_0^l F_n^{(k-1)} \left[\exp\left[\int_0^s F_n'\, dt\right]\right]^{k-2} ds\Big|_{n=0} = 0 \tag{7}$$

will be a necessary condition for this stability.

Corollary. If on the closed integral curve condition (7) is satisfied and $F_n^{(k)}$ (for even k) does not vanish identically and does not change sign, then this curve is a semistable limit cycle; if, however, on this curve condition (7) is satisfied and $F_n^{(k)}$ (for odd k) does not vanish identically, remaining constantly < 0 (> 0), then this curve will be a stable (unstable) limit cycle.

The result obtained is more general than the known Poincaré criteria for stability and instability ([3]):

$$\oint_0^l (P'_x + Q'_y)\,dt \begin{cases} < 0 & \text{stability,} \\ > 0 & \text{instability} \end{cases}$$

for the system

$$\frac{dy}{dt} = P(x, y), \qquad \frac{dx}{dt} = Q(x, y),$$

where l is the period of the periodic solution, and the integral is taken along L in the direction of increasing t.

3. Let us now consider some results relating to the study of one special case of limit cycles, the so-called regular cycles ([1]). It turns out that in this case conditions (5) and (6) of Theorem 1 are simplified.

Theorem 2. For stability (instability) of a regular limit cycle L it is sufficient that the condition

$$F'_n\big|_{n=0} = 0, \dots, F_n^{(k-1)}\Big|_{n=0} = 0, \qquad F_n^{(k)}\Big|_{n=0} < 0\ (> 0) \tag{8}$$

hold for odd k, and for semistability

$$F'_n\big|_{n=0} = 0, \dots, F_n^{(k-1)}\Big|_{n=0} = 0, \qquad F_n^{(k)}\Big|_{n=0} \neq 0 \tag{9}$$

for even k.

Thus, the question of the stability of a regular cycle is resolved without the integral form of the main theorem, which is convenient for practice, since the derivatives are computed without difficulty.

Conditions (8) and (9) are at the same time also conditions for the regularity of the limit cycle, so that we have found a criterion for recognizing regular cycles.

An example of an irregular limit cycle is given in paper [2].

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Note: Figure translations are in progress. See original paper for figures.