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Abstract

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Turbulent Viscosity in a Shallow Sea Due to Wave Motion

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As is known, wave motion decays with depth, as a result of which there is a velocity gradient in the vertical direction. The resulting velocity pulsations lead to the wandering of vortices from one layer to another and thereby to turbulent exchange.

S. V. Dobroklonskii ([1]) determined the coefficient of turbulent exchange due to wave motion for the case of a deep sea, proceeding from the equations of the trochoidal theory of waves and using the basic relations of the semi-empirical theory of turbulence.

In a shallow sea, wave motion gives rise to vortices whose dimensions may be comparable with the depth, which does not permit the use of the theory of locally isotropic turbulence. Therefore we shall also make use of the relations of the semi-empirical theory of turbulence, but, unlike Dobroklonskii, we shall start from the equations of the theory of waves at finite fluid depth, taking into account that the orbits along which the fluid particles move in the wave are not circular but elliptical in form:

$$x = x_0 + \frac{h}{2}\frac{\operatorname{ch} k(H - z_0)}{\operatorname{sh} kH} \sin k(ct - x_0),$$

$$z = z_0 - \frac{h}{2}\frac{\operatorname{sh} k(H - z_0)}{\operatorname{sh} kH} \cos k(ct - x_0), \tag{1}$$

where h is the wave height, and H is the depth of the fluid.

Denoting $h/2 \operatorname{sh} kH = \theta$ and introducing the new variable $x_1 = ct - x_0$, we write, instead of (1):

$$x = ct - x_1 + \theta \operatorname{ch} k(H - z_0) \sin kx_1,$$

$$z = z_0 - \theta \operatorname{sh} k(H - z_0) \cos kx_1. \tag{2}$$

The transition from (1) to (2) by means of the new variable x_1 physically means that the entire mass of fluid is, as it were, imparted a velocity equal to $-c$, while the structure of the field of velocity pulsations is not changed.

Let us assume the sea bottom to be "mirror-smooth," i.e., we shall not take into account the turbulization arising from bottom roughness, nor the additional turbulization arising as a result of the breaking of wave crests in shallow water ([2]).

In accordance with Prandtl' s theory, let us introduce the so-called "mixing length" l, averaged with respect to x_1 over the period length of the motion λ,

$$l = \varkappa \frac{\overline{(\partial u/\partial z)}_{x_1}}{\overline{(\partial^2 u/\partial z^2)}_{x_1}}. \quad (3)$$

and the coefficient of turbulent exchange of momentum in the vertical direction according to Kármán, averaging it in the same way:

$$A_v = \rho l^2 \left(\overline{\frac{\partial u}{\partial z}}\right)_{x_1}. \quad (4)$$

Since, according to the theory of turbulent viscosity, the viscosity coefficient is equal to the coefficient of turbulent exchange A_v, the problem is thus reduced to finding from (2) the values of $\partial u/\partial z$ and $\partial^2 u/\partial z^2$, with their subsequent averaging with respect to x_1 over the period length of the motion λ, i.e., to finding a certain integral value of these derivatives over the wavelength λ. Having first determined $u = dx/dt = kc\theta \operatorname{ch} k(H - z_0) \cos kx_1$, denoting for brevity $k(H - z_0) = \alpha$, $kx_1 = \beta$, and taking into account that $u = f(x, z_0)$, we find the values of the first and second derivatives:

$$\frac{\partial u}{\partial z} = -k^2 c\theta \frac{\operatorname{sh}\alpha \cos\beta - k\theta \operatorname{sh}\alpha \operatorname{ch}\alpha}{1 - k^2\theta^2(\cos^2\beta + \operatorname{sh}^2\alpha)},$$

$$\begin{aligned}\frac{\partial^2 u}{\partial z^2} = &-\frac{k^2 c\theta}{\left[1 - k^2\theta^2(\cos^2\beta + \operatorname{sh}^2\alpha)\right]^3}\Big\{\big[(-k \operatorname{ch}\alpha \cos\beta + k^2\theta \operatorname{ch}^2\alpha + k^2\theta \operatorname{sh}^2\alpha) \times \\ &\times (1 - k^2\theta^2\cos^2\beta - k^2\theta^2\operatorname{sh}^2\alpha) - (\operatorname{sh}\alpha\cos\beta - k\theta \operatorname{sh}\alpha \operatorname{ch}\alpha)\, 2k^3\theta^2 \operatorname{sh}\alpha \operatorname{ch}\alpha\big] \times \\ &\times (k\theta \operatorname{ch}\alpha \cos\beta - 1) + \big[-k \operatorname{sh}\alpha \sin\beta(1 - k^2\theta^2\operatorname{sh}^2\alpha) \\ &- (\operatorname{sh}\alpha\cos\beta - k\theta \operatorname{sh}\alpha \operatorname{ch}\alpha)\, 2k^3\theta^2 \cos\beta \sin\beta\big] k\theta \operatorname{sh}\alpha \operatorname{ch}\alpha\Big\}.\end{aligned}$$

Computing the integrals $\frac{1}{\lambda}\int_{x_1=0}^{x_1=\lambda} \frac{\partial u}{\partial z}\, dx_1$ and $\frac{1}{\lambda}\int_{x_1=0}^{x_1=\lambda} \frac{\partial^2 u}{\partial z^2}\, dx_1$, we obtain the averaged values of these derivatives:

$$\left(\overline{\frac{\partial u}{\partial z}}\right)_{x_1} = \frac{ck^3\theta^2 \operatorname{sh}\alpha \operatorname{ch}\alpha}{\sqrt{(1-k^2\theta^2 \operatorname{sh}^2\alpha)(1-k^2\theta^2 \operatorname{ch}^2\alpha)}}, \tag{5}$$

$$\begin{aligned}\left(\overline{\frac{\partial^2 u}{\partial z^2}}\right)_{x_1} = {} & \frac{c\pi k^3\theta^2}{2\lambda\sqrt{(1-k^2\theta^2 \operatorname{sh}^2\alpha)(1-k^2\theta^2 \operatorname{ch}^2\alpha)}}\Big[6(\operatorname{ch}^2\alpha + \operatorname{sh}^2\alpha) \\ & - k^2\theta^2(24\operatorname{sh}^2\alpha \operatorname{ch}^2\alpha + 8\operatorname{ch}^2\alpha - 9\operatorname{sh}^2\alpha) + k^4\theta^4(12\operatorname{sh}^4\alpha \operatorname{ch}^2\alpha \\ & + 7\operatorname{sh}^2\alpha \operatorname{ch}^2\alpha + 2\operatorname{ch}^2\alpha + 3\operatorname{sh}^2\alpha) + k^6\theta^6(3\operatorname{sh}^2\alpha \operatorname{ch}^2\alpha - 2\operatorname{sh}^2\alpha \operatorname{ch}^4\alpha)\Big].\end{aligned} \tag{6}$$

Substituting (5) and (6) into (3) and (4), we obtain:

$$\begin{aligned}A_v = {} & \frac{\varkappa^2\pi\rho h^2}{T}\frac{\operatorname{sh}^3 2\alpha}{\operatorname{sh}^2 kH}\sqrt{(1-k^2\theta^2 \operatorname{sh}^2\alpha)^7(1-k^2\theta^2 \operatorname{ch}^2\alpha)^7} \\ & \times \Big[6(\operatorname{ch}^2\alpha + \operatorname{sh}^2\alpha) - k^2\theta^2(24\operatorname{sh}^2\alpha \operatorname{ch}^2\alpha + 8\operatorname{ch}^2\alpha - 9\operatorname{sh}^2\alpha) \\ & + k^4\theta^4(12\operatorname{sh}^4\alpha \operatorname{ch}^2\alpha + 7\operatorname{sh}^2\alpha \operatorname{ch}^2\alpha + 2\operatorname{ch}^2\alpha + 3\operatorname{sh}^2\alpha) \\ & + k^6\theta^6(3\operatorname{sh}^2\alpha \operatorname{ch}^2\alpha - 2\operatorname{sh}^2\alpha \operatorname{ch}^4\alpha)\Big]^{-2},\end{aligned} \tag{7}$$

where ρ is the density of water, and $\varkappa$ is the "universal" constant, taken from pipe experiments as equal to 0.36–0.40.

Estimating the terms entering into (7), and neglecting very small terms, we obtain, with sufficient accuracy, that

$$A_v \approx \frac{\varkappa^2\pi\rho h^2}{36T}\frac{\operatorname{sh}^3 2k(H-z_0)}{\operatorname{sh}^2 kH \operatorname{ch}^2 2k(H-z_0)}. \tag{8}$$

Assuming, in the particular case, that the sea is deep, i.e. $H=\infty$, and taking into account that the hyperbolic sine of infinity may be taken equal to the hyperbolic cosine of infinity, it is not difficult to obtain from (8) Dobroklonskii' s formula for the deep sea

$$A_v \approx \frac{\chi^2\pi\rho h^2}{18T}e^{-4\pi z_0/\lambda}. \tag{9}$$

Let us investigate, with the aid of formula (8), the dependence of the exchange coefficient A_v on the variation, in the horizontal direction, of the dimensionless depth H/λ. Taking the first derivative of (8) with respect to the depth H and setting it equal to zero, we find that the exchange coefficient A_v has its maximum value at $H/\lambda = 0.1$. Figure 1 gives the graph of the dependence $A_v = f(H/\lambda)$

Fig. 1

Figure 1: Fig. 1

for a wave of height $h = 100$ cm, with period $T = 2.5$ sec., at $Z_0 = 0$, i.e. for the sea surface. From this graph it is seen that, beginning with $H/\lambda = 0.5$, the exchange coefficient A_v does not depend on the depth of the sea, and therefore its value may be determined from Dobroklonskii' s formula (9).

Fig. 1

For values $H/\lambda < 0.5$ the exchange coefficient A_v increases rapidly, reaching a maximum at $H/\lambda = 0.1$. It is of interest that the relative depth $H/\lambda = 0.1$ corresponds to the depth at which wave destruction (breaking) occurs, and physically a discontinuity of the function should occur here. It is not difficult to calculate that at this depth the horizontal axis of the elliptical orbit along which the fluid particles move in the wave becomes exactly twice as large as the vertical axis. This, apparently, leads to a sharp increase in the horizontal component of velocity, and consequently also in the vertical gradient of the horizontal velocity du/dz, and thereby to a significant change in the structure of the velocity field in the wave. All this is reflected in an increase of the exchange coefficient of the quantity of motion A_v in the vertical direction. It is possible that such an intense increase of the exchange coefficient A_v with decreasing relative depth H/λ is one of the causes producing wave breaking.

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CITED LITERATURE

1. S. V. Dobroklonskii, DAN, **58**, No. 7, 1345 (1947).

2. V. V. Shuleikin, Transactions of the Moscow Hydrophysical Institute, Academy of Sciences of the USSR, **9** (1956).

Note: Figure translations are in progress. See original paper for figures.