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Abstract

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MATHEMATICS

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ON METHODS OF ANALYSIS OF CERTAIN EXTREMAL PLANNING-PRODUCTION PROBLEMS

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In studying questions connected with the formulation of a rational plan that ensures the best use of resources and the maximum output of the required products, methods for analyzing mathematical models of the indicated problems should prove of substantial benefit.

Let there be l types of final products, m types of intermediate products, and n types of production factors. There are N technological methods of production. Each method $\pi_s (s = 1, \dots, N)$ is characterized by three vectors

$$X^{(s)}(x_1^{(s)}, \dots, x_l^{(s)}), \quad Y^{(s)}(y_1^{(s)}, \dots, y_m^{(s)}), \quad Z^{(s)}(z_1^{(s)}, \dots, z_n^{(s)}), \tag{1}$$

whose components show the volume of production of final products, intermediate products, and production factors, respectively (negative components mean expenditures).

Admissible plans are defined as positive linear combinations of the basic methods, i.e., a plan P is defined by the vector $P(p_1, \dots, p_N)$ $(p_s \geqslant 0)$. Under the assumption of linearity, which we adopt, the plan P is characterized by the vectors

$$X = \sum_{s=1}^{N} p_s X^{(s)}, \quad Y = \sum_{s=1}^{N} p_s Y^{(s)}, \quad Z = \sum_{s=1}^{N} p_s Z^{(s)}. \tag{2}$$

We consider the finding of a plan under the following conditions:

1) The expenditures of production factors are bounded by the vector $-Z_0$ $(Z_0 > 0)$,

$$\sum p_s Z^{(s)} \geqslant -Z_0 \quad \left(\sum p_s z_k^{(s)} \geqslant -z_0^{(k)};\ k = 1, 2, \dots, n\right). \tag{3}$$

2) There is no expenditure of intermediate products in the plan as a whole,

$$\sum p_s Y^{(s)} \geqslant 0. \tag{4}$$

3) The volume of final output, taking into account the specified assortment X_0 ($X_0 > 0$), attains a maximum, i.e.

$$\sum p_s X^{(s)} \geqslant k X_0, \tag{5}$$

where k has the greatest possible value.

A plan $\bar{P} = (\bar{p}_1, \dots, \bar{p}_N)$ satisfying conditions 1)–3) is called **optimal**.

Theorem. Suppose the following condition is fulfilled: the relations

$$\sum p_s X^{(s)} \geqslant 0, \qquad \sum p_s Y^{(s)} \geqslant 0, \qquad \sum p_s Z^{(s)} \geqslant 0, \qquad p_s \geqslant 0, \tag{6}$$

are possible only when $p_1 = \dots = p_N = 0$. Then an optimal plan $\overline{P} = (\bar{p}_1, \dots, \bar{p}_N)$ exists, and to it there corresponds such a system of multipliers (valuations) for all types of products and production factors

$$\Xi = (\xi_1, \dots, \xi_l), \qquad |H| = (\eta_1, \dots, \eta_m), \qquad Z = (\zeta_1, \dots, \zeta_n); \qquad \Xi,\ |H|,\ Z \geqslant 0,$$

that

$$(\Xi, X^{(s)}) + (|H|, Y^{(s)}) + (Z, Z^{(s)}) =$$

$$= \sum_{i=1}^{l} \xi_i x_i^{(s)} + \sum_{j=1}^{m} \eta_j y_j^{(s)} + \sum_{k=1}^{n} \zeta_k z_k^{(s)} \leqslant 0 \qquad (s = 1, \dots, N), \tag{7}$$

$$(\Xi, X^{(s)}) + (|H|, Y^{(s)}) + (Z, Z^{(s)}) = 0, \quad \text{if } \bar{p}_s > 0. \tag{8}$$

Conversely, if for some plan $\overline{P}$, satisfying the conditions $\overline{Z} = -Z_0$, $\overline{Y} = 0$, $\overline{X} = kX_0$, there are multipliers such that conditions (7), (8) are fulfilled, then this plan is optimal (if the weaker conditions $\overline{Z} \geqslant -Z_0$, $\overline{X} \geqslant kX_0$, $\overline{Y} \geqslant 0$ are fulfilled, then what has been said is true if, for the components for which the inequality sign occurs, the corresponding multipliers are equal to zero).

Proof. Put $\sup k = k_0$ ($k_0 \leqslant \infty$). For $k_\nu \to k_0$, $k_\nu < k_0$, there will be plans $P^{(\nu)} = (p_1^{(\nu)}, \dots, p_N^{(\nu)})$ such that

$$\sum p_s^{(\nu)} Z^{(s)} \geqslant -Z_0, \qquad \sum p_s^{(\nu)} Y^{(s)} \geqslant 0, \qquad \sum p_s^{(\nu)} X^{(s)} \geqslant k_\nu X_0.$$

Let $\sum |p_s^{(\nu)}| = \sigma_\nu$. It is impossible that $\overline{\lim}\, \sigma_\nu = +\infty$. Otherwise, passing to a subsequence, we would obtain

$$\sigma_\nu^{-1} p_s^{(\nu)} \to \tilde{p}_s$$

and for the plan $\widetilde{P} = (\tilde{p}_1, \dots, \tilde{p}_N)$

$$\sum \tilde{p}_s Z^{(s)} \geqslant 0, \qquad \sum \tilde{p}_s Y^{(s)} \geqslant 0, \qquad \sum \tilde{p}_s X^{(s)} \geqslant 0; \qquad \sum \tilde{p}_s = 1,$$

which contradicts the condition. Thus σ_ν are bounded; it is then clear that $k_0 = \sup k < +\infty$. Passing to a subsequence, we may assume that $\lim p_s^{(\nu)} = \bar{p}_s$; then

$$\sum \bar{p}_s Z^{(s)} \geqslant -Z_0, \qquad \sum \bar{p}_s Y^{(s)} \geqslant 0, \qquad \sum \bar{p}_s X^{(s)} \geqslant k_0 X_0,$$

i.e. the plan $\overline{P}(\bar{p}_1, \dots, \bar{p}_N)$ is optimal.

Next consider, in the $(l + m + n)$-dimensional space, the set K of vectors $U = (X, Y, Z)$ corresponding to all admissible plans. K is a polyhedral convex cone. The point $(\overline{X}, \overline{Y}, \overline{Z})$, corresponding to the optimal plan, lies on its boundary, since the positive hyperoctant of the space, translated to the point $(\overline{X}, \overline{Y}, \overline{Z})$, contains no interior points of the cone.

Consider the hyperplane H separating this hyperoctant and the cone K. It has an equation of the form

$$(\Xi, X) + (|H|, Y) + (Z, Z) = 0.$$

The coefficients of this equation—nonnegative numbers ξ_i, η_j, ζ_k—will be the required multipliers. Since, for all methods, the corresponding vectors belong to the cone, these vectors lie on one side of H, i.e.

$$(\Xi, X^{(s)}) + (|H|, Y^{(s)}) + (Z, Z^{(s)}) \leqslant 0.$$

Further we have

$$\sum \bar{p}_s \big[(\Xi, X^{(s)}) + (|H|, Y^{(s)}) + (Z, Z^{(s)}) \big] = (\Xi, \overline{X}) + (|H|, \overline{Y}) + (Z, \overline{Z}) \leqslant 0,$$

whence it is clear that

$$(\Xi, X^{(s)}) + (|H|, Y^{(s)}) + (Z, Z^{(s)}) = 0$$

when $\bar{p}_s > 0$. The last assertion of the theorem is verified without difficulty.

Remark 1. The condition imposed in the theorem is natural. Namely, if for a plan satisfying (6) it should turn out-offsetof

$\sum \tilde{p}_i > 0$ and $|\tilde{X}| + |\tilde{Y}| + |\tilde{Z}| > 0$, then this would mean that some types of products or factors can be produced without cost. Obviously, they should have been excluded from consideration. If, however, $\sum \tilde{p}_i > 0$ and $\tilde{X} = 0$, $\tilde{Y} = 0$, $\tilde{Z} = 0$, then one of the types of products or factors can be expressed through the others.

Remark 2. The conditions determining an optimal plan may be taken in another form as well. For example, one may require that the volume of production of certain products be fixed in the assignment, or the composition and volume of output may be completely specified, while it is required to obtain the minimum cost of all kinds, or of a specified kind.

The arguments given extend to all similar cases. For the existence of a plan, only the compactness of the set of plans satisfying the imposed restrictions is important; and for the existence of multipliers, it is important that the state be extremal (that there be a ray issuing from the desired point which certainly does not belong to the interior of the set K).

Let us also note that if no preliminary conditions are imposed, and a certain plan is given that is conditionally optimal, in the sense that no change of it is possible under which the volumes of production for all kinds of products would increase and all kinds of expenditures of production factors would decrease, then for such a plan there exist multipliers of the type indicated in the theorem.

Remark 3. In the particular case where intermediate products are absent, and in each method only one type of product figures (or only one type of production factor), then in order to characterize a plan it is sufficient to introduce estimates only for products (or, respectively, only for factors).

Remark 4. The results of considering problems A, B, C in work ([1]) are consequences of this theorem. A special case of this theorem, with more restrictive conditions (the Leontief model), is given in ([2]).

For the effective solution of problems of constructing an optimal plan, various methods may be used, based on the application of "resolving multipliers," which give a characterization of the maximal plan.

First of all, if there is a certain plan satisfying conditions (3), (4), (5), then, in order to verify its extremality, it is sufficient to check the possibility of determining multipliers from conditions (8), with (7) and the conditions of equality to zero of some of the multipliers being satisfied. If such a plan is not given in advance, then for the simultaneous finding of an optimal plan and a system of multipliers several processes of successive approximations may be applied:

1) **Successive improvement of the plan.** Starting from some plan satisfying conditions (3), (4), (5), we determine multipliers from conditions (8); if conditions (7) turn out not to be fulfilled, then a method is found whose adjoining makes it possible to increase the value k ([3],[4]) and ([5], Ch. I, §7).

In other words, we approach the point $(\bar{X}, \bar{Y}, \bar{Z})$ gradually from inside the cone. A similar process, but without the use of resolving multipliers, is used in the so-called simplex method of Dantzig ([6]) in linear programming.

2) **Approximation by conditionally optimal plans.** Having assigned a certain system of multipliers, we choose, in accordance with (7) and (8), the methods to be included in the plan, and construct it with (3), (4) taken into account. Then we modify the multipliers, successively increasing the

value of k (motion along the surface of the cone; see ([1]) and ([5]), Ch. I, §6). An improvement of this device is the algorithm of the method of the extreme point ([7]).

3) **Approximation to the system of multipliers, and also to the plan, with the introduction of two-sided bounds for the values of the multipliers** (see ([5]), Ch. I, §8).

The enumerated methods for finding a plan with the use of "resolving multipliers" (estimates) prove effective even in very complicated cases. In addition to facilitating the finding of the optimal plan, the use–

tion of multipliers makes it possible to solve a number of other questions: correction of the plan when conditions change, assessment of the expediency of using methods not taken into account when drawing up the plan, etc.

With the aid of the problem considered above, a more complex problem can also be analyzed, when the plan is considered not for one, but for a series of moments of time $t = 1, 2, \dots, T$. Considering each type of output and of factors at different moments as an independent type of output or factors, we characterize the technological methods $\bar{r}_s$ already by means of three matrices: $X^{(s)} = \|x_{i,t}^{(s)}\|$; $Y^{(s)} = \|y_{j,t}^{(s)}\|$; $Z^{(s)} = \|z_{k,t}^{(s)}\|$ $(i = 1, 2, \dots, l;\ j = 1, 2, \dots, m;\ k = 1, 2, \dots, n;\ s = 1, 2, \dots, N;\ t = 1, 2, \dots, T)$.

The concept of an "optimal plan" can be introduced here in various ways; for example, one may require that the output always be not below a prescribed level, and that its time average have a given composition and the greatest possible value. We shall not dwell on any of such conditions, but shall confine ourselves to a characterization of a conditionally optimal plan (cf. Remark 3). On the basis of the theorem proved, we conclude that to every conditionally optimal plan there will correspond matrices of multipliers $\Xi = \|\xi_{i,t}\|$, $\mathrm{H} = \|\eta_{j,t}\|$, $Z = \|\zeta_{i,t}\|$, such that conditions similar to (7), (8) are fulfilled. Naturally these multipliers (estimates) can be normalized. For example, putting $\xi_{i,t} = \lambda_t \xi'_{i,t}, \dots$, we can achieve fulfillment of the condition

$$\sum_i \xi'_{i,t} = 1 \quad (t = 1, 2, \dots, T),$$

i.e., so that the set of final output has value 1. However, then conditions (7) and (8) will already take another form; for example, (7) will be replaced by

$$\sum_t \lambda_t \left(\sum_i \xi'_{i,t} x_{i,t}^{(s)} + \sum_j \eta'_{j,t} y_{j,t}^{(s)} + \sum_k \zeta'_{k,t} z_{k,t}^{(s)} \right) \leq 0, \tag{7'}$$

i.e., in evaluating production methods, output and costs incurred at different moments must be reduced to one moment by means of the multipliers λ_t. The quantities λ_t (more correctly λ_t^{-1}) represent a general characteristic of the growth of output over time for the given plan. In addition to their evident economic meaning, isolating them is very useful in searching for an optimal plan. Namely, it is expedient to apply the following process of successive approximations. Having specified certain methods involving a number of moments, then determine estimates for each moment, proceeding from the obtained assignments for the composition of output and costs, taking into account here methods that include only elements of the given moment. Then, on the basis of the estimates obtained, find λ_t and evaluate methods including elements from different times. Using some of them, improve the plan, and so on.

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Note: Figure translations are in progress. See original paper for figures.