Physical Chemistry

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Abstract

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DIFFUSIONAL DEPOSITION OF AEROSOLS ON A CYLINDER IN A FLOW AT SMALL CAPTURE COEFFICIENTS

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1. The calculation of the number of aerosol particles deposited from a viscous flow, owing to diffusion, on a cylinder perpendicular to the flow was carried out approximately by Langmuir (^{1}), who assumed that, when the cylinder is flowed around, all particles from the volume bounded by a certain streamline ψ_1 (ψ is the stream function), passing at $\theta = \pi/2$ at a distance $r_0 - a = x_0$ from the cylinder surface, have time to diffuse to its surface (θ and r are polar coordinates, a is the radius of the cylinder). The value x_0 was calculated from the expression for the mean absolute Brownian displacement of a particle

$$x = \left(\frac{4}{\pi} Dt\right)^{1/2}, \tag{1}$$

in which x was defined as the root-mean-square distance from the cylinder surface during motion along the streamline ψ_1 between $\theta = 5\pi/6$ and $\theta = \pi/6$, and the quantity t as the time of this motion. Upon substituting into (1) the values of x and t, expressed in terms of x_0, a, and w by Lamb' s formulas for the velocity distribution at $r - a \ll a$, it was obtained that

$$\frac{x_0}{a} = \left(0.56 \frac{D}{wa}\right)^{1/3},$$

where

$$w = \frac{v_0}{2(2.00 - \ln \mathrm{Re})}, \qquad \mathrm{Re} = \frac{2av_0}{\nu},$$

and v_0 is the velocity far from the cylinder. The value of the capture coefficient ε given by Langmuir has the form

$$\varepsilon = \frac{\psi_1}{v_0 a} = \frac{w}{v_0}\left(2\frac{r_0}{a} \ln \frac{r_0}{a} + \frac{a}{r_0} - \frac{r_0}{a}\right),$$

where

$$\frac{r_0}{a} = 1 + \frac{x_0}{a}.$$

Since the quantity x_0 was calculated under the condition $r - a \ll a$, the rough approximate expression obtained, strictly speaking, applies only to the region of small capture coefficients and, because $\frac{x_0}{a} \ll 1$, becomes

$$\varepsilon = 2\frac{w}{v_0}\left(\frac{x_0}{a}\right)^2 = 1.36\left(\frac{w}{v_0}\right)^{1/3}\left(\frac{D}{v_0 a}\right)^{2/3}. \tag{2}$$

It is not difficult to show, however, that at small ε the formulas for diffusional deposition can be obtained rigorously from the differential equation of diffusion by the method developed by V. G. Levich for the solution of analogous problems (^{2}).

2. The equation of convective diffusion on a circular cylinder perpendicular to the flow has, in polar coordinates, the form

$$\frac{\partial c}{\partial t} + v_r\frac{\partial c}{\partial r} + \frac{v_\theta}{r}\frac{\partial c}{\partial \theta} = D\left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r} + \frac{1}{r^2}\frac{\partial^2 c}{\partial \theta^2}\right).$$

Upon reaching the steady state, assuming that the rate of diffusional transfer in the θ direction is considerably smaller than in the r direction or than the rate of convective transfer, we obtain

$$v_r\frac{\partial c}{\partial r} + \frac{v_\theta}{r}\frac{\partial c}{\partial \theta} = D\left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r}\right).$$

If we restrict ourselves to the region

$$\frac{r-a}{a} = \varkappa \ll 1,$$

i.e., to the case of small capture coefficients, then the diffusion equation takes the form

$$v_r\frac{\partial c}{\partial r} + \frac{v_\theta}{r}\frac{\partial c}{\partial \theta} = D\frac{\partial^2 c}{\partial r^2}.$$

Passing from the coordinates (r, θ) to the coordinates (ψ, θ) (^{2}) and taking into account that

$$v_\theta = -\frac{\partial \psi}{\partial r} \quad \text{and} \quad v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta},$$

we obtain, for $r \sim a$,

$$\left(\frac{\partial c}{\partial \theta}\right)_{\psi} = aD\frac{\partial}{\partial \psi}\left(v_{\theta}\frac{\partial c}{\partial \psi}\right)_{\theta} . \tag{3}$$

3. In the case of viscous flow around a cylinder, described by Lamb' s velocity distribution, for $\varkappa \ll 1$,

$$\psi = 2wa\varkappa^2 \sin\theta \quad \text{and} \quad v_{\theta} = -4w\varkappa \sin\theta .$$

Neglecting $\varkappa$, equation (3) takes the form

$$\frac{\partial c}{\partial \theta} = -\left(8waD^2 \sin\theta\right)^{1/2} \frac{\partial}{\partial \psi}\left(\psi^{1/2}\frac{\partial c}{\partial \psi}\right) . \tag{4}$$

Introducing the variable

$$\varphi = \int_{\theta}^{\theta_1} \left(8waD^2 \sin\theta\right)^{1/2} \, d\theta ,$$

we obtain

$$\frac{\partial c}{\partial \varphi} = \frac{\partial}{\partial \psi}\left(\psi^{1/2}\frac{\partial c}{\partial \psi}\right) . \tag{5}$$

The solution of (5) under the boundary conditions $c = 0$ at $\psi = 0$ (on the surface of the cylinder) and $c = c_0$ at $\psi = \infty$ (far from the cylinder) has the form (2)

$$c = \frac{3}{\Gamma(1/3)}\left(\frac{4}{9}\right)^{1/3} c_0 \int_0^u e^{-4/9\,u^3} \, du , \tag{6}$$

where

$$u = \psi^{1/2}/\varphi^{1/3} .$$

The diffusion flux per unit length of the cylinder is equal to

$$2J = 2\int_0^{\pi} D\left(\frac{\partial c}{\partial r}\right)_{r=a} a \, d\theta = 2D\int_0^{\pi}\left(\frac{\partial c}{\partial u}\frac{du}{\partial \psi}\frac{\partial \psi}{\partial r}\right)_{\psi=0} a \, d\theta . \tag{7}$$

Taking into account that in the incident stream (at $\theta = \pi$) $c = c_0$, we obtain from (6): at $\theta = \pi$, $u = \infty$ and $\varphi = 0$, i.e., $\theta_1 = \pi$. Therefore from (6) and (7) we obtain

$$2J = \frac{6}{\Gamma(1/3)} \left(\frac{4}{9}\right)^{1/3} c_0 (waD^2)^{1/3} \int_0^\pi \frac{\sin^{1/2}\theta\, d\theta}{\left(\int_\theta^\pi \sin^{1/2}\theta\, d\theta\right)^{1/3}}. \tag{8}$$

The integral appearing in the denominator, after the substitution $\sin^{1/2}\theta = z$, becomes, for $\theta < \frac{\pi}{2}$,

$$\int_0^1 \frac{2z^2\,dz}{\sqrt{1-z^4}} + \int_z^1 \frac{2z^2\,dz}{\sqrt{1-z^4}},$$

and for $\theta > \frac{\pi}{2}$,

$$\int_0^1 \frac{2z^2\,dz}{\sqrt{1-z^4}} - \int_z^1 \frac{2z^2\,dz}{\sqrt{1-z^4}}.$$

Since

$$\int_z^1 \frac{z^2\,dz}{\sqrt{1-z^4}} = \int_z^1 \sqrt{\frac{1+z^2}{1-z^2}}\,dz - \int_z^1 \frac{dz}{\sqrt{1-z^4}} = \sqrt{2}\,E\left(\arccos z, \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}} F\left(\arccos z, \frac{1}{\sqrt{2}}\right),$$

where E and F are elliptic integrals of the 2nd and 1st ···

then the values of the integral in the denominator of formula (8) can be obtained from tables. Using these values, and calculating the value of the entire integral in formula (8) graphically, we obtain

$$2J = \frac{6}{\Gamma(1/3)} \left(\frac{4}{9}\right)^{1/3} 2.71\, c_0 (waD^2)^{1/3}.$$

Hence the capture coefficient ε is equal to

$$\varepsilon = \frac{J}{v_0 a c_0} = 2.32 \left(\frac{w}{v_0}\right)^{1/3} \left(\frac{D}{v_0 a}\right)^{2/3}, \tag{9}$$

which differs substantially, in the magnitude of the numerical factor, from Langmuir' s expression (2).

The derivation of formula (9) is valid only for point particles, since for particles of finite size the boundary condition on the surface of the cylinder changes and, in addition, the influence of the interception effect must be taken into account (a particle moving along a streamline is deposited even in the absence of diffusion when its center approaches the cylinder to a distance $r - a \leq R$, where R is the radius of the particle). In the absence of diffusion, the capture coefficient due to interception for viscous flow at $\frac{R}{a} = \varkappa_0 \ll 1$ is equal to $2\frac{w}{u_0}\varkappa_0^2$. Thus, expression (9) is valid when

$$\left(\frac{D}{v_0 a}\right)^{2/3} = \frac{1}{\mathrm{Pe}^{2/3}} \ll 1$$

and

$$\left(\frac{R^3}{a^3}\frac{v_0 a}{D}\right)^{2/3} \ll 1,$$

where Pe is the Peclet number.

4. In the case of potential flow past a cylinder, for $\varkappa \ll 1$

$$\psi = 2v_0 a x \sin\theta \quad \text{and} \quad v_\theta = -2v_0 \sin\theta.$$

Equation (3) takes the form

$$\frac{\partial c}{\partial \theta} = -\,2v_0 a D \sin\theta\,\frac{\partial^2 c}{\partial \psi^2},$$

which can be transformed into

$$\frac{\partial c}{\partial(\cos\theta + 1)} = 2v_0 a D\,\frac{\partial^2 c}{\partial \psi^2}. \tag{10}$$

The boundary conditions of the problem, written in the form (10), are as follows: $c = 0$ at $\psi = 0$ and $c = c_0$ at $\cos\theta + 1 = 0$ ($\theta = \pi$). Under these conditions the solution of (10) has the form

$$c = c_0 \Phi\left(\frac{\psi}{\sqrt{8v_0 a D(\cos\theta + 1)}}\right), \tag{11}$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy$$

is Kramp’ s function.

Analogously to (7), we obtain

$$2J = 2\int_0^\pi \left(D \left(\frac{\partial c}{\partial \psi} \right)_{\psi=0} \right) \left(\frac{\partial \psi}{\partial r} \right)_{r=a} a\, d\theta = c_0 \left(\frac{8 v_0 a D}{\pi} \right)^{1/2} \int_0^\pi \frac{\sin\theta\, d\theta}{(\cos\theta + 1)^{1/2}} = 8 c_0 \left(\frac{v_0 a D}{\pi} \right)^{1/2},$$

which corresponds to Boussinesq' s solution ([3]) in the analogous problem of heat transfer.

The capture coefficient will be equal to

$$\varepsilon = \frac{J}{v_0 a c_0} = \frac{4}{\sqrt{\pi}} \left(\frac{D}{v_0 a} \right)^{1/2}. \tag{12}$$

Since the capture coefficient for pure interception in potential flow for $\chi_0 \ll 1$ is equal to $2\chi_0$, expression (12) is valid under the conditions

$$\left(\frac{D}{v_0 a} \right)^{1/2} = \frac{1}{\mathrm{Pe}^{1/2}} \ll 1, \qquad \left(\frac{R^2}{a^2} \frac{v_0 a}{D} \right)^{1/2} \ll 1.$$

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[2] V. G. Levich, *Physicochemical Hydrodynamics*, Moscow, 1952, Ch. 2, §§ 13–14.
[3] Ph. Frank, R. Mises, *Differential and Integral Equations of Mathematical Physics*, Leningrad-Moscow, Part 2, Ch. 14, § 2 (1), 1937.

Note: Figure translations are in progress. See original paper for figures.