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MATHEMATICS

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ON THE SPECTRAL PROPERTIES OF SELF-ADJOINT ELLIPTIC OPERATORS

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In the author's note ([1]) the boundedness of the eigenfunctions of a certain class of self-adjoint operators was established. The condition ensuring the boundedness of the eigenfunctions consisted in the fact that, for at least one λ_0, the resolvent of the operator R_{λ_0} is an integral operator with kernel $K(x, y, \lambda_0)$, for which the inequality

$$\int_{-\infty}^{\infty} \left|K^2(x, y, \lambda_0)\right| \, dy < C, \tag{1}$$

holds, where the constant C does not depend on x. It was also stated there that if $q(x) > a > -\infty$, then the resolvent of the operator $-\Delta + q$ satisfies inequality (1). In the present note inequality (1) is established for a sufficiently broad class of elliptic operators, and one proposition concerning the index of defect of such operators is also established.

1. Let us consider the operator L in the space of N-dimensional vector-functions

$$L = \sum_{k_1 + \dots + k_n = 2m} A^{k_1 \cdots k_n}(x) \frac{\partial^{2m}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} + T, \tag{2}$$

where $x = (x_1, \dots, x_n)$, $-\infty < x_i < \infty$; $A^{k_1 \cdots k_n}(x)$ is the matrix $\|a_{ij}^{k_1 \cdots k_n}(x)\|$, $i, j = 1, 2, \dots, N$. By T is denoted a linear differential operator of order $< 2m$.

We shall assume that the following conditions are satisfied:

a) The coefficients $a_{ij}^{k_1 \cdots k_n}(x)$ have $2m$ continuous derivatives, bounded in the whole space R_n. The coefficients at the lower derivatives have first-order derivatives, and they are bounded in all of R_n together with these derivatives.

b) The matrices $A^{k_1 \cdots k_n}(x)$ are symmetric. The operator L is formally symmetric in the sense of differential expressions.

c) The characteristic roots $\lambda_i(s, x)$ of the matrix

$$\sum_k A^{k_1 \cdots k_n}(x)\, s_1^{k_1} \cdots s_n^{k_n}$$

for real s and $|s| = s_1^2 + \cdots + s_n^2 = 1$, and for any $x \in R_n$, satisfy the inequalities $\lambda_i(s, x) < -\delta$, where $\delta > 0$ and does not depend on s and x.

We note that if, for the system of equations, the matrices at the highest derivatives $A^{k_1 \cdots k_n}(x)$ are symmetric and property (c) is satisfied, then the system will be strongly elliptic. As the domain of definition D_L

of the operator L take the totality of all finite vector-functions $f(x) = \{f_1(x), \ldots, f_N(x)\}$ having $2m$ continuous derivatives. By the assumptions made, the operator L on D_L will be a symmetric differential operator. For such systems one can state the following assertions.

Theorem 1. *If the operator L satisfies conditions (a), (b), (c) and is semi-bounded on D_L, then there exists a real number λ_0 such that for the kernel of the resolvent $K(x, y, \lambda_0)$ the inequality (1) holds.*

Theorem 2. *If the operator L satisfies conditions (a), (b), (c) and is semi-bounded on D_L, then it has deficiency index* $(0, 0)$.

1. For the proof of Theorems 1 and 2, consider the Cauchy problem for the parabolic system

$$\partial u/\partial t = Lu - \lambda u, \qquad u(x, 0) = \varphi(x), \tag{3}$$

where $\varphi(x) \in D_L$. It is known that the Cauchy problem (3), under the assumptions made concerning the coefficients, has a unique solution ([2]). Let M be a constant bounding the coefficients and their derivatives throughout the space R_n.

Lemma 1. *The Green function of problem (3), for sufficiently large $\lambda > 0$, satisfies the estimate*

$$|G(x, \xi, t)| \le \frac{A}{t^{n/2m}} e^{-B|x-\xi|^{2m'}/t^{1/(2m-1)}} e^{-at}, \tag{4}$$

where $1/2m' + 1/2m = 1$, *a depends on* λ *and* M; A *and* B *do not depend on* t, x, ξ.

The validity of the lemma follows from the fact that after the substitution $u = e^{\lambda t} v$, the Cauchy problem (3) becomes the Cauchy problem for the equation

$$\partial v/\partial t = Lv,$$

whose Green function satisfies the inequality

$$|G^*(x,\xi,t)| \le \frac{e^{Mt}}{t^{n/2m}} e^{-B|x-\xi|^{2m'}/t^{1/(2m-1)}}.$$

Here M is a constant majorizing the coefficients. Performing the inverse substitution and taking $\lambda > M$, we obtain the required assertion. Inequality (4) shows that

$$\int_0^\infty G(x,\xi,t)\,dt$$

converges absolutely. We shall use this fact.

Let us now prove Theorem 2. Suppose that the deficiency index of the operator L is not $(0,0)$. Then, by semiboundedness, the operator L will have at least two distinct semibounded self-adjoint extensions ([3]). It is known that the resolvent R_λ corresponding to any such extension is generated by an integral operator with Carleman kernel

$$R_\lambda f(x) = \int_{-\infty}^{\infty} K(x,y,\lambda) f(y)\,dy.$$

The resolvent kernel $K(x,y,\lambda)$ corresponding to the resolvent R_λ can be expressed through the spectral kernel $\Psi(x,y,\lambda)$:

$$K(x,y,\lambda) = \int_\lambda^\infty \frac{\Psi(x,y,\lambda)}{\eta - \lambda}\,d\sigma(\eta).$$

With the aid of the kernel $\Psi(x,y,\lambda)$ we form the function

$$v(x,t) = \int_\lambda^\infty e^{(\lambda-\lambda_0)t} \left\{ \int_R \Psi(x,y,\lambda)\varphi(y)\,dy \right\} d\sigma(\lambda),$$

where λ_0 is such that $(L\varphi - \lambda_0\varphi, \varphi) \le 0$, $\varphi \in D_L$. It is easy to verify that $v(x,t)$ is a solution of the Cauchy problem

$$\partial u/\partial t = Lu - \lambda_0 u, \qquad u(x,0) = \varphi(x).$$

This follows from the fact that $L_x\Psi(x,y,\lambda) = \lambda\Psi(x,y,\lambda)$, $x \neq y$. Since the solution of the Cauchy problem is unique, we have

$$\int_\lambda e^{(\lambda-\lambda_0)t}\left\{\int_R \Psi(x,y,\lambda)\varphi(y)\,dy\right\}d\sigma(\lambda) = \int_R G(x,y,t)\varphi(y)\,dy. \tag{5}$$

Integrating equality (5) with respect to t from 0 to ∞, we obtain

$$K(x,y,\lambda_0) = \int_\lambda \frac{\Psi(x,y,\lambda)}{\lambda-\lambda_0}\,d\sigma(\lambda) = \int_0^\infty G(x,y,t)\,dt.$$

Thus we see that the kernel $K(x,y,\lambda)$, and together with it the resolvent and the extension itself, are determined uniquely, which contradicts the fact that the deficiency index is not $(0,0)$.

This theorem is a strengthening of the corresponding theorems of M. A. Naimark and I. M. Rapoport, who considered the case of ordinary equations, assuming that the coefficients are either summable or have a limit at infinity ([4]).

Let us note that the Cauchy problem for the wave equation, in establishing the deficiency index of the operator $-\Delta + q$, was first used by A. Ya. Povzner ([5]). Starting from inequality (4), it is easy to see that the following lemma holds.

Lemma 2. If $|x-y| \geq 1$, then

$$\left|\int_0^\infty G(x,y,t)\,dt\right| < Ce^{-a|x-y|}.$$

Thus, for $|x-y| \geq 1$, the resolvent kernel satisfies the inequality

$$|K(x,y,\lambda_0)| \leq Ce^{-a(x-y)}.$$

As for the behavior of the kernel in a neighborhood of the values $x = y$, it is well known that the kernel has a singularity of the type of a fundamental solution.

3. Denote by E_λ the resolution of the identity of the operator L. Let $g^{(\alpha)}$ be generating vectors and let $\sigma_\alpha(\lambda) = (E_\lambda g^{(\alpha)}, g^{(\alpha)})$ be spectral measures. It is known that the functions $dE_\lambda g^{(\alpha)}/d\sigma_\alpha(\lambda)$ are eigenfunctions and form a complete system ([6,7]). Since inequality (1) has been proved, taking Theorem 1 into account, one may assert that the following theorem is valid.

Theorem 3. If the operator satisfies conditions (a), (b), (c) and is semibounded, then almost all, with respect to the measure $\sigma_\alpha(\lambda)$, eigenfunctions $dE_\lambda g^{(\alpha)}/d\sigma_\alpha(\chi)$ are bounded in x. In the aggregate of the variables x, y, for almost all λ with respect to the spectral measure $\sigma(\lambda)$, the spectral kernel $\Psi(x, y, \lambda)$ will also be bounded.

Let us note that an individual eigenfunction can be made to grow arbitrarily fast.

4. If the coefficients of the operator L are not bounded in the whole space, then for general elliptic systems estimates of type (1) cannot be obtained. It is probable, however, that for semibounded operators estimates of type (1) are nevertheless true. Let us consider one such example. Suppose an operator is given

$-\Delta + q(x)$ throughout R_3. We shall assume that the operator is semibounded; however, we do not assume that the coefficient $q(x)$ is bounded below.

Theorem 4. *Suppose that for every function $f(x)$ in the domain of definition of the self-adjoint operator $L = -\Delta + q(x)$, $|q| < M(r)$, the inequality*

$$(Lf, f) \geq a(f, f).$$

is satisfied. Then, for some λ_0, the kernel of the resolvent $K(x, y, \lambda_0)$ admits the estimate

$$\int_{-\infty}^{\infty} |K^2(x, y, \lambda_0)|\, dy < CM(r), \qquad r = |x|.$$

It is also assumed that $M(x + y) \leq M(x) + M(y)$.

Proof. Denote by $\varphi(x, y, \lambda)$ the kernel of the resolvent of the operator $-\Delta + \lambda^2$. Then $K - \varphi$ has no singularities. Since

$$-\Delta_x K(x, y, \lambda) + q(x)K(x, y, \lambda) + \lambda^2 K(x, y, \lambda) = 0,$$

$$-\Delta_x \varphi(x, y, \lambda) + \lambda^2 \varphi(x, y, \lambda) = 0,$$

it follows that

$$-\Delta_x(K - \varphi) + \lambda^2(K - \varphi) + q(K - \varphi) = -q(x)\varphi. \tag{6}$$

Multiplying (6) scalarly by $K - \varphi$, we obtain

$$\int_{-\infty}^{\infty} L(K-\varphi)(K-\varphi)\,dx + \lambda^2(K-\varphi, K-\varphi) = \int_{-\infty}^{\infty} q(x)\varphi(K-\varphi)\,dx. \quad (7)$$

Without loss of generality one may assume that $(L\psi, \psi) \geq 0$. Equality (7) therefore gives

$$\int_{-\infty}^{\infty} K^2(x,y,\lambda)\,dy \leq \sqrt{\frac{A}{\lambda}} + a,$$

where

$$A = \left(\int (q\varphi)^2\,dy\right)^{1/2}, \qquad a = \int_{-\infty}^{\infty} \varphi^2(x,y,\lambda)\,dy.$$

But A grows in the same way as $q(x)$, in view of the fact that

$$\varphi(x,y,\lambda) = \frac{e^{-\lambda(x-y)}}{|x-y|}, \qquad M(x+y) \leq M(x) + M(y).$$

One can proceed analogously in the case of an arbitrary elliptic equation.

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