SovietRxiv

Soviet-era science, translated into English

Corresponding Member of the Academy of Sciences of the USSR A. A. MARKOV

1957

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-195701.18383

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR A. A. MARKOV

ON THE INVERSION COMPLEXITY OF SYSTEMS OF FUNCTIONS

1. Let n be a natural number different from zero. By the alphabet of formulas in n variables we shall mean the alphabet $\{0, 1, x_1, \dots, x_n, \&, \vee, \neg, (,)\}$. We shall denote this alphabet by Φ_n. The letters $x_1, \dots, x_n$ of the alphabet Φ_n will be called variables, and the letters 0 and 1—constants.

By formulas in n variables we shall mean words in the alphabet Φ_n defined by the following generating rules.

F1. The one-letter words

$$0, 1, x_1, \dots, x_n$$

are formulas in n variables.

F2. If the words P and Q are formulas in n variables, then the words $(P \& Q)$ and $(P \vee Q)$ are formulas in n variables.

F3. If the word P is a formula in n variables, then the word $\neg P$ is a formula in n variables.

Formulas beginning with the letter $\neg$ will be called negative formulas. Every formula occurring in a formula P will be called a subformula of the formula P. Every subformula of at least one of the formulas $P_1, \dots, P_m$ will be called a subformula of the system of formulas $P_1, \dots, P_m$.

The number of distinct negative subformulas of the system of formulas $P_1, \dots, P_m$ will be called the inversion complexity of this system of formulas.

2. Every formula in n variables defines, in the known way, a Boolean function of n arguments, i.e., a function taking only the values 0 and 1 on n arguments that take only these same two values. Every system of m formulas in n variables defines a system of m Boolean functions of n arguments (m is a positive integer).

It is known that, conversely, every Boolean function of n arguments can be defined by a formula in n variables. Therefore every system of m Boolean functions of n arguments can be defined by a system of m formulas in n variables.

SovietRxiv

Suppose we have a system of m Boolean functions $f_1, \dots, f_m$ of n arguments. It can be defined by various systems of formulas in n variables. Each such system of formulas has a definite inversion complexity. The least of the inversion complexities of systems of formulas defining the system of functions $f_1, \dots, f_m$ will be called the inversion complexity of the system of functions $f_1, \dots, f_m$. In particular, the least of the inversion complexities of formulas defining the function f will be called the inversion complexity of the function f. We shall denote the inversion complexity of the system of functions $f_1, \dots, f_m$ by the symbol $\mathrm{Inv}(f_1, \dots, f_m)$. A special case of this notation is the notation $\mathrm{Inv}(f)$ for the inversion complexity of the function f.

3. A system of n constants $(a_1, \dots, a_n)$ will be called an n-dimensional Boolean vector. Every Boolean function of n argu-

can be regarded as a function of an n-dimensional Boolean vector, taking the values 0 and 1.

Let A and B be n-dimensional Boolean vectors. Let $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$. We shall say that A **precedes** B if $a_i \le b_i$ $(i = 1, \dots, n)$ and $a_j < b_j$ for some j. In what follows, the notation $A < B$ will mean that the Boolean vector A precedes the Boolean vector B.

Let f be a Boolean function of n arguments; $A_1, \dots, A_r$ a sequence of n-dimensional Boolean vectors. We shall say that this sequence is an **alternating chain of the function** f, if

$$A_i < A_{i+1} \qquad (1 \le i < r),$$

$$f(A_i) = \begin{cases} 1 & \text{for odd } i, \\ 0 & \text{for even } i \end{cases} \qquad (1 \le i \le r).$$

The number r will then be called the **length** of the alternating chain $A_1, \dots, A_r$.

If the function f is not identically equal to zero, then by the **alternation** of the function f we shall mean the greatest of the lengths of the alternating chains of this function. To the function identically equal to zero we assign alternation 0.

We shall denote the alternation of the function f by the symbol $\mathrm{Alt}(f)$. It is clear that

$$0 \le \mathrm{Alt}(f) \le n + 1$$

for every Boolean function f of n arguments.

4. Let r be a natural number. By the **binary size** of the number r we shall mean the least of the natural numbers y such that

SovietRxiv

$$r < 2^y.$$

We shall denote the binary size of the number r by the symbol $D(r)$. It is clear that $D(0) = 0$, while $D(r) = [\lg_2 r] + 1$ for $r > 0$. It is also clear that for $r > 0$, $D(r)$ is the number of digits in the binary representation of the number r.

The symbol $\mathrm{pd}(r)$ will henceforth denote the natural number immediately preceding r, for $r > 0$, and the number 0 for $r = 0$ (cf. (1), p. 200).

5. The following theorem holds, giving an expression for the inversion complexity of a Boolean function f in terms of the alternation of this function.

5.1. *For every Boolean function f the equality holds*

$$\mathrm{Inv}(f) = \mathrm{pd}(D(\mathrm{Alt}(f))).$$

6. Let m be a positive integer. There are exactly 2^{m2^n} distinct systems of m Boolean functions of n arguments. We shall denote the greatest of their inversion complexities by $I(n, m)$. The following theorems give a simple way of computing the thus defined arithmetic function I of two arguments.

6.1. $I(n, 1) = \mathrm{pd}(D(n + 1))$.

6.2. $I(n, m) = D(n)$ for $m > 1$.

7. The following lemma plays an essential role in the proof of these results.

7.1. Let k be a positive integer and $n = 2^k - 1$. Define Boolean functions s_t $(t = 0, \dots, n)$ of n arguments by the equalities

$$s_0(x_1, \dots, x_n) = 1,$$

$$s_t(x_1, \dots, x_n) = \bigvee_{1 \le i_1 < \dots < i_t \le n} x_{i_1} \& \cdots \& x_{i_t} \qquad (1 \le t \le n).$$

For every k-tuple $\varepsilon_1, \dots, \varepsilon_k$ $(\varepsilon_i = 0, 1$ for $i = 1, \dots, k)$ define the Boolean function $S_{\varepsilon_1, \dots, \varepsilon_k}$ of n arguments by the equality

$$S_{\varepsilon_1, \dots, \varepsilon_k}(x_1, \dots, x_n) = S_t(x_1, \dots, x_n),$$

where

$$t = \sum_{i=1}^{k} \varepsilon_i 2^{k-i}.$$

Define the Boolean function g_1 of n arguments by the equality

SovietRxiv

$$g_1(x_1, \dots, x_n) = S_{1,0,\dots,0}(x_1, \dots, x_n);$$

for $j = 2, \dots, k$ define the Boolean function g_j of $n + j - 1$ arguments by the equality

$$g_j(x_i, \dots, x_n, y_1, \dots, y_{j-1}) = \\ = \bigvee_{\varepsilon_1, \dots, \varepsilon_{j-1}} S_{\varepsilon_1, \dots, \varepsilon_{j-1}, 1, 0, \dots, 0}(x_1, \dots, x_n) \& y_1^{1-\varepsilon_1} \& \dots \& y_{j-1}^{1-\varepsilon_{j-1}},$$

where y^0 denotes 1, and y^1 denotes y. Define the Boolean function g of $n + k$ arguments by the equality

$$g(x_1, \dots, x_n, y_1, \dots, y_k) = \bigvee_{\varepsilon_1, \dots, \varepsilon_k} S_{\varepsilon_1, \dots, \varepsilon_k}(x_1, \dots, x_n) \& y_1^{1-\varepsilon_1} \& \dots \& y_k^{1-\varepsilon_k}.$$

Define successively the functions h_j ($j = 1, \dots, k$) of n arguments by the equalities

$$h_1(x_1, \dots, x_n) = \neg g_1(x_1, \dots, x_n),$$

$$h_j(x_1, \dots, x_n) = \neg g_j(x_1, \dots, x_n, h_1(x_1, \dots, x_n), \dots, h_{j-1}(x_1, \dots, x_n))$$

$$(1 < j \leq k).$$

Then the identities

$$g(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n, h_1(x_1, \dots, x_n), \dots, h_k(x_1, \dots, x_n)) = \neg x_i$$

$$(1 \leq i \leq n),$$

$$g_k(x_1, \dots, x_n, h_1(x_1, \dots, x_n), \dots, h_{k-1}(x_1, \dots, x_n)) = \sum_{i=1}^{n} x_i,$$

hold, where the summation sign denotes summation modulo 2.

SovietRxiv

8. The question of the inversion complexity of a single Boolean function was studied by E. N. Gilbert [2]. However, he did not obtain result 5.1, although he was apparently close to obtaining it. To the best of our knowledge, this author did not study the inversion complexity of systems of Boolean functions.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
17 V 1957

REFERENCES

[1] S. K. Kleene, *Introduction to Metamathematics*, Moscow, 1957. [2] E. N. Gilbert, *J. of Math. and Physics*, **33**, 1, 57 (1954).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.