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1957

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Abstract

Full Text

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THE METHOD OF GRIDS FOR EQUATIONS OF S. L. SOBOLEV TYPE

(Presented by Academician S. L. Sobolev on 16 I 1957)

Let the function $u(x,t)$ be a solution in the domain $Q = \Omega \times [\Omega, l]$ of the equation

$$Lu \equiv \frac{\partial^2}{\partial t^2} L_0 u + \frac{\partial}{\partial t} L_1 u + L_2 u = f(x,t), \tag{1}$$

satisfying the conditions

$$u\Big|_{t=0} = \varphi(x), \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x), \tag{2}$$

$$u\Big|_S = 0, \tag{3}$$

where $x = (x_1, x_2, \dots, x_n)$; S is the boundary of the domain Ω;

$$L_0 u \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}\right) + a_0 u,$$

$$L_s u \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(B_{ij}^s\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i^s \frac{\partial u}{\partial x_i} + b_0^s u, \qquad s = 1, 2.$$

For simplicity of exposition we shall restrict ourselves to the case when the coefficients of equation (1) depend only on x, are bounded and, in addition,

$$-\sum_{i,j=1}^{n} A_{ij}\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2, \qquad \alpha = \text{const} > 0, \qquad a_0 \geq 0.$$

The existence and uniqueness of a generalized solution of the problem posed have been proved in papers ([1,2]); therefore we shall dwell on the convergence of solutions of the finite-difference analogue of equation (1) to the generalized solution and on the differential properties of the generalized solution in a closed domain. In what follows we shall adhere to the notation of paper ([3]).

We shall call a function u, belonging together with $\partial u/\partial t$ to the class $\overset{0}{D}_1(Q)$ and satisfying the identity

$$\iint_Q \left[\sum_{i,j=1}^{n} \left(A_{ij} \frac{\partial^3 u}{\partial x_j \partial t^2} + B^1_{ij} \frac{\partial^2 u}{\partial x_j \partial t} + B^2_{ij} \frac{\partial u}{\partial x_j} \right) \frac{\partial \Phi}{\partial x_j} + \right.$$

$$\left. + \left(\sum_{i=1}^{n} b^1_i \left(\frac{\partial^2 u}{\partial x_i \partial t} + b^2_i \frac{\partial u}{\partial x_i} \right) + a_0 \frac{\partial^2 u}{\partial t^2} + b^1_0 \frac{\partial u}{\partial t} + b^2_0 u - f \right) \Phi \right] dQ = 0 \qquad (4)$$

for any Φ from $\overset{0}{L}_1(Q)$, if, moreover,

$$\int_\Omega \left(\frac{\partial u(x, \Delta t)}{\partial t} - \psi \right)^2 + (u(x, \Delta t) - \varphi)^2 \, d\Omega \to 0 \qquad \text{as } \Delta t \to 0.$$

Construct a finite-difference analogue of equation (1). Partition the space $R_{n+1}(x_1, x_2, \dots, x_n, t)$ by the planes $x_i = k_i h$, $t = k_0 \Delta t$, $h > 0$, $\Delta t > 0$, $i = 1, 2, \dots, n$, where k_i are integers, into parallelepipeds $Q_{k_1 \dots k_n k_0}$, the coordinates of whose points satisfy the inequalities $k_i h \le x_i \le (k_i + 1)h$, $i = 1, 2, \dots, n$, $k_0 \Delta t \le t \le (k_0 + 1)\Delta t$. Points with coordinates $(k_1 h, \dots, k_{nh}, k_0 \Delta t)$ shall be called vertices or mesh points. Denote by Ω_h the domain composed of those cubes $\Omega_{k_1 \dots k_n}$ $(k_i h \le x_i \le (k_i + 1)h,\ i = 1, 2, \dots, n)$ which belong to the domain Ω, and by Q_h the prism $\Omega_h \times [0, m\Delta t]$, where $m = [l/\Delta t]$. The boundary surface of Ω_h will be denoted by S_h, and $S_h \times [0, m\Delta t] = F_h$.

Replace equation (1) at the mesh points Q_h by the difference equations

$$L_{hu} \equiv [L_{0h} u]_{\bar{t}t} + [L_{1h} u]^0_t + L_{2h} u = f(x, t), \qquad (5)$$

where

$$L_{0h} u \equiv \sum_{i,j=1}^{n} (A_{ij} u_{x_j})_{\bar{x}_i} + a_0 u,$$

$$L_{sh} u \equiv \sum_{i,j=1}^{n} (B^s_{ij} u^s_{x_j})_{\bar{x}_i} + \sum_{i=1}^{n} b^s_i u^s_{x_i} + b^s_0 u, \qquad s = 1, 2,$$

$$u^0_t = \frac{1}{2}(u_t + u_{\bar{t}}).$$

Define at the points Q a function u_h satisfying inside Q_h equations (5), and for $t = 0$ and $t = \Delta t$ the initial conditions

$$u_h(k_1h,\dots,k_{nh},0)=\varphi(k_1h,\dots,k_{nh}),$$

$$u_h(k_1h,\dots,k_{nh},\Delta t)=\varphi(k_1h,\dots,k_{nh})+\Delta t\,\psi(k_1h,\dots,k_{nh}),$$

and equal to zero at the points of F_h and outside Q_h. The solvability of the system obtained is a consequence of the fundamental inequalities derived below. In what follows, the passage from the functions f,φ,ψ to the averaged functions f_h,φ_h,ψ_h, needed in certain limiting passages, is omitted; the functions u_h are denoted by u.

Sum the equality $u_t^0(L_{hu}-f)=0$, valid for all mesh points when $\Delta t\le t\le(m-1)\Delta t$, over all points of the prism $\Omega_h\times[\Delta t,(p-1)\Delta t]$ $(p<m)$. Then, omitting the summation sign over i,j, we obtain

$$\Delta t\sum_{\Delta t}^{(p-1)\Delta t}h^n\sum_{\Omega_h}u_t^0(L_{hu}-f)=0; \tag{6}$$

but

$$\Delta t\sum_{\Delta t}^{(p-1)\Delta t}h^n\sum_{\Omega_h}u_t^0[L_{0h}u]_{\bar t t}=\left\{h^n\sum_{\Omega_h}-A_{ij}u_{x_j\bar t}u_{x_i\bar t}+\frac{a_0}{2}u_{\bar t}^2\right\}_{\Delta t}^{p\Delta t},$$

and from (6), for sufficiently small Δt, we obtain the inequality

$$h^n\sum_{\Omega_h}(u_{x_i\bar t})^2\Bigg|_{p\Delta t}\le C_1e^{C_2p\Delta t}\left[h^n\sum_{\Omega_h}(\varphi_{x_i}^2+\psi_{x_i}^2)+\Delta t\sum_{\Delta t}^{p\Delta t}h^n\sum_{\Omega_h}f^2\right]. \tag{7}$$

Considering now the equality

$$\Delta t\sum_{\Delta t}^{(p-1)\Delta t}h^n\sum_{\Omega_h}u_{\bar t t}(L_{hu}-f)=0,$$

we obtain from it, using inequality (7), the inequality

$$\Delta t\sum_{\Delta t}^{(p-1)(\Delta t)}h^n\sum_{\Omega_h}(u_{\bar t t x_i})^2\Bigg|_{p\Delta t}\le$$

$$\leq C_3 e^{C_4 p\Delta t}\left[h^n\sum_{\Omega_h}(\varphi_{x_i}^2+\psi_{x_i}^2)+\Delta t\sum_{\Delta t}^{p\Delta t}h^n\sum_{\Omega_h}f^2\right]. \tag{8}$$

Carrying out further arguments coinciding with those given in Chapter III of [3], we establish that the sequence $\{u_h\}$, together with the sequences $\{u_{ht}\}$, as $\Delta t, h \to 0$, converges weakly in $W_2^{(1)}(Q)$ to functions $u, \partial u/\partial t$, from $\overset{\circ}{D}_1(Q)$; $u(x,t)$ is a generalized solution of our problem; moreover, $\{u_{\bar{t}}\}$ and $\{u_{ht}\}$ converge, respectively, to u and $\partial u/\partial t$ on each plane $t = \text{const}$ in the norm $L_2(\Omega)$, uniformly with respect to $t \in [0,l]$; for the limiting function u the inequality

$$\int_0^t\int_\Omega\sum_{i=1}^n\left[\left(\frac{\partial^2 u}{\partial x_i\partial t}\right)^2+\left(\frac{\partial^3 u}{\partial x_i\partial t^2}\right)^2\right]d\Omega\,dt\leq$$

$$\leq C_5 e^{C_6 t}\left\{\int_\Omega\sum_{i=1}^n\left[\left(\frac{\partial\psi}{\partial x_i}\right)^2+\left(\frac{\partial\psi}{\partial x_i}\right)^2\right]d\Omega+\int_0^t\int_\Omega f^2\,d\Omega\,dt\right\}. \tag{9}$$

Let us investigate the differential properties of the solution obtained and determine under what conditions, imposed on the data functions in the problem and on the smoothness of the boundary S of the domain Ω, the generalized solution found belongs to $W_2^{(k)}(Q)$. We assume that the domain Ω can be covered by a finite number of overlapping canonical domains $\Omega_1, \dots, \Omega_N$ [3] in such a way that the sum $\Omega^1, \dots, \Omega^N$ gives all of Ω. Introduce in $\overline{\Omega}$ new coordinates $y_1, y_2, \dots, y_n$ by means of functions $y_i = y_i(x_1, x_2, \dots, x_n)$ $(i = 1, 2, \dots, n)$, $k+1$ times continuously differentiable in $\overline{\Omega}$. Let the canonical domain Ω_1 be mapped onto the cube D_1, and the whole domain Ω onto a domain D; let

$$J=\frac{D(x_1,x_2,\dots,x_n)}{D(y_1,y_2,\dots,y_n)}>0\quad\text{in }\overline{\Omega}.$$

Since $L_0 u$ is a self-adjoint operator, in the new coordinates $y_1, y_2, \dots, y_n$ the self-adjoint operator will be $L_0' u = J L_0 u$. Taking this into account, instead of equation (1) we consider in the new coordinates the equation

$$JLu = Jf. \tag{10}$$

This equation is of the same form as equation (1), and therefore we shall retain for it the former notation of coefficients and independent variables. For the solutions of the difference analogue of equation (10), the estimates (7), (8) will be valid. Suppose that the coefficients of equation (1) have continuous derivatives up to order $k-1$ in the cylinder $\overline{Q}$; $f \in W_2^{(k-1)}(Q)$; $\varphi, \psi \in W_2^{(k)}(\Omega)$; the boundary functions $z_n = \omega(z_1, z_2, \dots, z_{n-1})$ are assumed continuously differentiable with respect to $z_1, z_2, \dots, z_{n-1}$ up to order $k+1$. Then the difference quotients with respect to t of the solution of (10) are easily estimated, since the function u_t satisfies the equation

$$L_n u_t = f_t$$

and the conditions

$$u_t\big|_{F_h} = 0, \qquad u_t\big|_{t=0} = \psi, \qquad u_{\bar{t}t}\big|_{t=0} = \psi_1.$$

Having estimated u_t, we estimate u_{tt}, u_{ttt}, etc., up to order $k+1$.

We now estimate the other difference quotients. For this purpose we introduce into consideration, as in § 3 of Ch. III of the work [(3)], the domains D', D'', D''', D_1 and the function ζ, and also, as in the derivation of inequalities (7), (8), by the method developed in [(3)], from the equalities

$$(L_h u)_{x_k} = f_{x_k}, \qquad (L_{hu})_{\bar{x}_k} = f_{\bar{x}_k}$$

we obtain that

$$\Delta t \sum_{\Delta t}^{(p-1)\Delta t} h^n \sum_{D'} \sum_{i,j=1}^{n} \left(u_{x_j x_{it}}\right)^2 + \left(u_{x_j \bar{x}_i tt}\right)^2 < C_7.$$

By the same method one proves the boundedness of sums of the form

$$\Delta t \sum_{\Delta t}^{(p-1)\Delta t} h^n \sum_{D_1} \left\{ u_{ht} + \sum_{s=1}^{m} \sum_{\alpha_0, \dots, \alpha_s = 0}^{n} \left(\frac{\Delta^s u_{ht}}{\Delta t^{\alpha_0} \Delta x_{\alpha_1} \cdots \Delta x_{\alpha_s}} \right)^2 \right\}, \qquad m = 3, \dots, k.$$

As $\Delta t, h \to 0$, $u_{ht} \to \partial u / \partial t$ weakly in $W_2^{(k)}(Q)$, and for the function u we obtain the theorem:

Theorem. Suppose that the coefficients of equation (1) have continuous derivatives up to order $k-1$ in the cylinder $\bar{Q}$; $f \in W_2^{(k-1)}(Q)$; $\varphi, \psi \in W_2^{(k)}(\Omega)$; the boundary S of the domain Ω is continuously differentiable $k+1$ times and $u|_S = 0$.

Then the generalized solution u of the mixed problem for equation (1) exists and, together with $\partial u / \partial t$, belongs to the space $W_2^{(k)}(Q)$. For it the inequality

$$\left\| \frac{\partial u}{\partial t} \right\|_{W_2^{(k)}(Q)} \le C_8 \left(\|\varphi\|_{W_2^{(k)}(\Omega)} + \|\psi\|_{W_2^{(k)}(\Omega)} + \|f\|_{W_2^{(k-1)}(Q)} \right)$$

holds.

This theorem makes it possible to establish differential properties of the solutions of the system

$$\frac{\partial \bar{U}}{\partial t} = A\bar{U} - \operatorname{grad} p + \bar{F}, \qquad \operatorname{div} \bar{U} = 0,$$

which was considered by the author in [4], for the case when the matrix A is constant and $p|_S = 0$ (for notation see [4]), since in this case the function p satisfies an equation of type (1), while

$$\left\| \frac{\partial \bar{U}}{\partial t} \right\|_{W_2^{(k)}(Q)} \leq C_9 \left(\|\bar{U}_0\|_{W_2^{(k)}(\Omega)} + \|\bar{F}\|_{W_2^{(k)}(Q)} + \|\operatorname{grad} p\|_{W_2^{(k)}(Q)} \right).$$

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Received
11 I 1957

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Note: Figure translations are in progress. See original paper for figures.