Yu. S. Bogdanov

1957

SovietRxiv

Abstract

Full Text

Yu. S. Bogdanov

LYAPUNOV NORMS IN LINEAR SPACES

(Presented by Academician V. I. Smirnov on 6 X 1956)

MATHEMATICS

1°. Notation. Let us denote: A —a linear set, a —elements of A; T —an automorphism of A; B —an n-dimensional subspace of A; b —elements of B; $\beta \equiv (b^1, b^2, \dots, b^n)$ —a basis of B; Δ —an ordered set; λ —a mapping of A into Δ; c —a real number; $C \equiv (c^i_j)$ —an n-dimensional matrix (i —column number, j —row number).

2°. Definition of a Lyapunov norm. We shall say that λ normizes A in the sense of Lyapunov if, for any a, a', c:

1) $\lambda(ca) \preceq \lambda(a)$;
2) $\lambda(a + a') \preceq \max\{\lambda a, \lambda a'\}$.

The element $\lambda a \in \Delta$ will be called the Lyapunov norm λ (or the λ-norm) of the element $a \in A$.

3°. Properties of the Lyapunov norm. From the definition of the Lyapunov norm it follows:

a) $c \neq 0 \to \lambda(ca) = \lambda a$;
b) $\lambda a \succeq \lambda 0$;
c) $\lambda(c_1 a^1 + c_2 a^2 + \dots + c_m a^m) =\preceq \max(\lambda a^1, \lambda a^2, \dots, \lambda a^m)$;
d) $c \neq 0,\ \lambda a \succ \lambda a^i\ (i = 1, 2, \dots, m) \to \lambda(ca + c_1 a^1 + \dots + c_m a^m) = \lambda a$;
e) $\lambda a^i \neq \lambda a^j,\ i \neq j,\ a^i \neq 0\ (i, j = 1, 2, \dots, m) \to a^1, a^2, \dots, a^m$ are linearly independent.

4°. The Lyapunov norm in a finite-dimensional linear space. Any m ($m > n$) elements of B are linearly dependent, and therefore (see 3°, e)) the set $\{\lambda b\}$, where b is any nonzero element of B, has no more than n pairwise distinct elements.

5°. Definition of a λ-basis. A basis β of the set B will be called a λ-basis if every linear combination of the elements of β has a λ-norm equal to the greatest of the λ-norms of those elements of β which enter into the combination with coefficients different from 0.

6°. Existence of a λ-basis. Let β be a basis of B. Then there exists a triangular matrix C with unit diagonal elements such that $\beta_1 = \beta_2 C$ is a λ-basis of B. The proof of the statement just made is carried out in the same way as the proof of Lyapunov' s theorem on the existence of a triangular matrix C transforming a given fundamental system of solutions of a system of homogeneous linear differential equations into a normal system of solutions ([1]).

7°. Properties of a λ-basis. Here and in the remaining paragraphs let β_1 denote a λ-basis of B; β_2 a basis of B, with any basis assumed to be written so that $\lambda b_i^j \preceq \lambda b_i^l$, if $j < l$ $(i, j = 1, 2, \dots, n)$. Then:

a) $0 \neq b \in B \to \lambda b \in \{\lambda b_1^1, \lambda b_1^2, \dots, \lambda b_1^n\}$;

b) $\lambda b \preceq \lambda b_1^i \to b = c_1 b_1^1 + c_2 b_1^2 + \dots + c_{i-1} b_1^{i-1}$;

c) in order that the basis β_2 be a λ-basis of B, it is necessary and sufficient that $\lambda b_2^i = \lambda b_1^i,\ i = 1, 2, \dots, n$.

8°. **The number of elements of a basis with one and the same λ-norm.** Let k denote the number of mutually distinct values λb_1^i $(i = 1, 2, \dots, n)$. Suppose that these values are $\delta_1, \delta_2, \dots, \delta_k$, with $\delta_i \succ \delta_j$ for $i < j$. Suppose further that n_i is the greatest possible number of linearly independent elements of B with Lyapunov norm not exceeding δ_i. Denote by $N_i(\beta_2)$ the number of elements of β_2 with λ-norm δ_i. Then:

a) $N_1(\beta_2) + N_2(\beta_2) + \dots + N_k(\beta_2) = n$;

b) $n_k = n,\ n_i < n_j$, if $i < j$;

c) $N_1(\beta_2) + N_2(\beta_2) + \dots + N_i(\beta_2) \le n_i$;

d) $N_1(\beta_1) = n_1,\ N_i(\beta_1) = n_i - n_{i-1}$ $(i = 2, 3, \dots, k)$.

9°. **Step matrices.** A matrix $C \equiv (c_j^i)$ will be called m_l; r-step if $c_j^i = 0$ for $i \le m_l < j$ for any $l = 1, 2, \dots, r$; $i, j = 1, 2, \dots, n$ $(m_r = n)$. A triangular matrix is, evidently, l; n-step. All nonsingular m_l; r-step matrices (for fixed m_l) form a group with respect to multiplication in the usual sense for matrices.

10°. **Transformation of a λ-basis.** The basis β_2, like any other system of n elements of B, can be represented in the form $\beta_2 = \beta_1 C$, where C is a real matrix. The basis β_2 will be a λ-basis if and only if C is an n_i; k-step nonsingular matrix. This can be proved by the same arguments by which the validity of the corresponding proposition is established for a normal system of solutions ([2]).

11°. **The sets M_i.** Denote by M_i the set of all elements of B with λ-norm δ_i, where 0 is not included in M_1, even if $\lambda 0 = \delta_1$. In addition, put $M_0 \equiv 0$. Then:

a) $M_i \cap M_j$ for $i \neq j$;

b) $M_0 \cup M_1 \cup \dots \cup M_k = B$.

12°. **The structure of M_i.** Each M_i is an n_i-dimensional hyperplane with the n_{i-1}-dimensional hyperplane removed. Indeed,

$$M_0 \cup M_1 \cup \dots \cup M_i = \bigcup \left(c_1 b_1^1 + c_2 b_1^2 + \dots + c_{n_i} b_1^{n_i}\right)$$

(the last sum is taken over all $c_1, c_2, \dots, c_{n_i} \in (-\infty, \infty)$). The converse assertion is also true: if B is represented as the sum of hyperplanes $P_0 \equiv 0;\ P_1, P_2, \dots, P_m$, each of which strictly contains the preceding ones, then one can indicate a Lyapunov norm λ such that the decomposition of the space B into the sets $M_1, M_2, \dots, M_k$ effected by it has the property: $k = m;\ M_i = P_i - P_{i-1}$ $(i = 1, 2, \dots, k)$.

13°. **λ-similar transformations of A.** An automorphism T of the linear set A: $TA = A$, will be called a λ-similar transformation if

$$\lambda a \preceq \max\{\lambda T a, \lambda T^{-1} a\}.$$

In order that T be a λ-similar transformation, it is necessary and sufficient that

$$\lambda a = \lambda T a = \lambda T^{-1} a$$

(a, as always, is any element of A).

14°. **Invariants of λ-similar transformations.** Under an automorphism T of the linear set A, an n-dimensional linear subset $B \subset A$ is transformed into an n-dimensional linear subset $\widetilde{B} \subset A$. It is not difficult to show (based, for example, on 7°, c)) that if T is a λ-similar transformation, then the λ-basis β_1 of the linear subset B under T passes into the λ-basis $\tilde{\beta}_1$ of the linear subset $\widetilde{B}$, and moreover $\lambda_1 b_1^i = \lambda \tilde{b}_1^i$ $(i = 1, 2, \dots, n)$. Thus the numbers n_i (see 8°) are invariants of all λ-similar transformations.

15°. **Examples.** The first example of a Lyapunov norm on the set of solutions of all possible systems of homogeneous linear differential equations specified for $t \geqslant t_0$ is the Lyapunov characteristic number ([3]), taken with the opposite sign. As a second example one may point to the aggregate (characteristic exponent, type of solution) considered by L. Markus ([4]). In this case the second $[\lambda'']$-norm refines the first $\{\lambda'\}$, i.e.,

$$\lambda'' a \supseteq \lambda'' a' \to \lambda' a \supseteq \lambda' a', \qquad \lambda'' a \subseteq \lambda'' a' \to \lambda' a = \lambda' a'.$$

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
3 X 1956

REFERENCES

[1] A. M. Lyapunov, *The General Problem of the Stability of Motion*, Moscow–Leningrad, 1950, pp. 48–50.
[2] Yu. S. Bogdanov, DAN, 57, No. 3, 215 (1947).
[3] A. M. Lyapunov, DAN, 57, No. 3, 39 (1947).
[4] L. Markus, Math. Zs., 62, 310 (1955).

Note: Figure translations are in progress. See original paper for figures.