



Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

YUAN CHZHAO-DIN

1957

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-195701.17962>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1957. Volume 117, No. 4

MATHEMATICS

YUAN CHZHAO-DIN

ON THE STABILITY OF DIFFERENCE SCHEMES FOR SOLVING DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE

(Presented by Academician S. L. Sobolev, 7 VI 1957)

In the approximate solution of boundary-value problems for differential equations of parabolic type by the grid method, one has to carry out a large amount of computational work. This is connected with the fact that, when explicit difference schemes are used, in order to ensure their stability one has to take the time step τ very small in comparison with the step h in the spatial coordinates ($\tau/h^2 \leq \text{const}$). If an implicit difference scheme is used, then in order to find the values of the solution on each layer one has to solve a system of linear algebraic equations.

In the present note several difference schemes are proposed, the calculation by which is stable for large τ , which gives a significant saving in the amount of computational work.

As indicated in the papers ^(1,2), the "first" eigen-elements of the difference operator L_h , approximating the differential operator L , converge to the eigen-elements of the operator L , whereas the "last" eigen-elements of the operator L_h have a parasitic character; but it is precisely they that often determine the stability of the computational process. In the schemes proposed by us, the influence of the last eigen-elements of L_h , which determine the instability of the schemes, is essentially suppressed.

Let us illustrate this by a simple example.

Let Q be a domain in the plane x, y with boundary Γ . Cover the plane x, y by a grid of straight lines parallel to the coordinate axes and spaced at distance h from one another. Denote by Q_h the set of grid nodes belonging to Q , and by Γ_h the set of nodes lying outside Q whose distance to Q is not greater than h . Consider the operator

$$\Delta_h u = u_{\bar{x}x} + u_{\bar{y}y} \text{ on } Q_h; \quad u|_{\Gamma_h} = 0,$$

where

$$u_x = \frac{1}{h}[u(x+h, y) - u(x, y)]; \quad u_{\bar{x}} = \frac{1}{h}[u(x, y) - u(x-h, y)]$$

and so on; Δ_h is a self-adjoint operator, for which there exists a complete orthonormal system of eigen-elements, and its eigenvalues λ , $\Delta_h u_\lambda + \lambda u_\lambda = 0$, are situated in the region

$$0 < \lambda \leq \Lambda = \frac{8}{h^2}. \quad (1)$$

Denoting by A a point of Q_h , and by B a point of Γ_h , let us consider the difference formulas

$$u(t, A) = \tau \Delta_h u(t - \tau, A) + u(t - \tau, A); \quad u(t, B) = 0,$$

$$u(t, A) = \frac{b + \tau \Delta_h}{b + 1} u(t, A) + \frac{1}{b + 1} u(t - \tau, A); \quad u(t, B) = 0, \quad (A)$$

which are, respectively, explicit and implicit difference schemes for the equation $\partial u / \partial t = \Delta u$ in the cylinder $[0 \leq t \leq T] \times Q$. We shall use the latter scheme for the iterations

$$u^{[k+1]}(t, A) = \frac{b + \tau \Delta_h}{b + 1} u^{[k]}(t, A) + \frac{1}{b + 1} u(t - \tau, A); \quad u^{[k+1]}(t, B) = 0. \quad (B(b))$$

Let λ be an eigenvalue, and $u_\lambda(A)$ the corresponding eigenelement of the operator Δ_h . Setting $u(t, A) = v_\lambda(t) u_\lambda(A)$ in (A), we obtain the equation for $v_\lambda(t)$

$$v_\lambda(t) = (1 - \tau \lambda) v_\lambda(t - \tau). \quad (A')$$

Setting $u^{[k]}(t, A) = v_\lambda^{[k]}(t) u_\lambda(A)$ in (B(b)), we obtain

$$v_\lambda^{[k+1]}(t) = \frac{b + \tau \lambda}{b + 1} v_\lambda^{[k]}(t) + \frac{1}{b + 1} v_\lambda(t - \tau),$$

or

$$v_\lambda^{[k]}(t) = \left[v_\lambda^{[0]}(t) - \frac{1}{1 + \tau \lambda} v_\lambda(t - \tau) \right] \left[\frac{b - \tau \lambda}{b + 1} \right]^k + \frac{1}{1 + \tau \lambda} v_\lambda(t - \tau). \quad (B')$$

We formulate the following computation scheme.

Computation scheme I_1 .

- a) Knowing the values $u(t - \tau, A)$, find the values $u(t, A)$ by scheme (A);
- b) taking the values found as the first approximation, perform one iteration by the scheme $(B(\tau\lambda - 1))$;
- c) on the basis of the values found, perform k iterations by the scheme $(B(\frac{1}{2}\tau\lambda))$.

Setting $u(t, A) = v_\lambda(t)u_\lambda(A)$, from (A') and (B') we obtain that, when scheme I_1 is applied, for $v_\lambda(t)$ we have the equation

$$v_\lambda(t) = \left\{ \frac{-\tau^2\lambda^2}{1 + \tau\lambda} \left[\frac{\tau\lambda - 1 - \tau\lambda}{\tau\lambda} \right] \left[\frac{\tau\lambda - 2\tau\lambda}{\tau\lambda + 2} \right]^k + \frac{1}{1 + \tau\lambda} \right\} v_\lambda(t - \tau) = \mu(\lambda, k)v_\lambda(t - \tau)$$

or

$$v_\lambda(t) = [\mu(\lambda, k)]^{t/\tau} v_\lambda(0),$$

where t/τ is an integer. Hence we conclude that if k is so large that

$$\max_{0 \leq \lambda \leq \Lambda} |\mu(\lambda, k)| \leq 1,$$

then scheme I_1 is stable in the norm

$$\|u\|_h^2 = h^2 \sum_{Q_h} u^2.$$

Assuming k to be an even number, for $\tau\Lambda = 8\tau/h^2 \rightarrow \infty$ we have the asymptotic lower bound k_1 of the values k satisfying the stability condition; namely, $4k_1/\tau\Lambda$ is the solution of the equation $z = 2e^{-1-z}$, whence

$$4k_1 \approx 0.46\tau\Lambda. \quad (2)$$

Comparing scheme I_1 with the ordinary explicit scheme, which uses only equation (A) and is stable under the condition $\tau'/h^2 \leq 1/4$, or $\tau'\Lambda \leq 2$, we find that, by this scheme, between two consecutive layers of scheme I_1 , separated from one another by the distance τ , it is necessary to compute $\tau/\tau' (\geq \frac{1}{2}\tau\Lambda)$ intermediate layers, whereas under scheme I_1 , to compute the solution on the layer t , it is necessary to carry out $k_1 + 2$ computations by formulas analogous to the formulas for computation on one layer by scheme (A). That is, when scheme I_1 is used and $\tau\Lambda \rightarrow \infty$, the computation is reduced by more than a factor of 4.3. Furthermore, in the right-hand side of formula $(B(\frac{1}{2}\tau\Lambda))$ the term $u^{[k]}(t, x)$ is absent, which makes it possible to carry out the iteration first for the nodes

layer t , the sum of whose indices is an even number, and then for the remaining nodes, then again for the nodes of the 1st set, etc. Owing to this the amount of computation is reduced almost by another factor of 2.

Carrying out the iteration after two steps, we obtain the following scheme:

Scheme I_2 .

- a) Knowing $u(t - 2\tau, A)$, by scheme (A) we compute the values on the layers $t - \tau$ and t ;
- b) using the values found, we perform 2 iterations by $(B(\tau\Lambda - 1))$ to refine $u(t, A)$;
- c) with the refined values we perform another $2k$ iterations by $(B(\frac{1}{2}\tau\Lambda))$.

The lower bound k_1 of the number k satisfying the stability condition for scheme I_2 , as $\tau\Lambda \rightarrow \infty$, satisfies the equation

$$\max_{1 < \alpha} \left\{ \frac{1}{2} \alpha (\alpha - 1) e^{-z} e^{-\alpha z} \right\} = 1, \quad \text{where } z = \frac{4k_1}{\tau\Lambda}.$$

For k_1 we have the asymptotic estimate

$$0.35\tau\Lambda \leq 4k_1 \leq 0.42\tau\Lambda. \quad (3)$$

As $\tau\Lambda \rightarrow \infty$, the amount of computational work when using scheme I_2 is reduced by more than a factor of 4.7 in comparison with the ordinary explicit scheme (A).

Carrying out iterations after more than two steps leads to unstable schemes.

Remark 1. All the arguments are valid for all cases considered in paper (2), where the operator Δ is approximated by the operator

$$\Delta_h = \frac{1}{Mh^2}(S_h - 1),$$

S_h being defined in § 1 of paper (2). Then

$$\Lambda = \frac{1}{Mh^2}(1 - \lambda(S));$$

$\lambda(S)$ is the lower bound of the spectrum of S_h for all h . Estimates (2), (3) for k_1 are preserved, and the amount of computational work is reduced by the above-indicated number of times in comparison with the ordinary explicit scheme.

In the case where the grid is parallelogram-shaped and S_h is a homogeneous operator, as $\tau \rightarrow 0$ the solutions obtained by the schemes under consideration converge uniformly for $0 \leq t \leq T$, $A \in \bar{Q}$ to the solution of the problem

$$\frac{\partial u}{\partial t} = \Delta u; \quad u|_{\Gamma} = 0; \quad u|_{t=0} = \varphi(A),$$

if $\varphi(A)$ is continuously differentiable with respect to x and y four times in \overline{Q} , and $\varphi(A)$ and its derivatives up to and including the 3rd order are equal to zero on Γ .

Remark 2. Everything said in Remark 1 also holds for the case when the number n of spatial coordinates is arbitrary; uniform convergence has been proved for $n = 1, 2, 3, 4, 5$.

Remark 3. If, just as in work (3), the operator

$$Lu = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left[a_i(t, x_1, \dots, x_n) \frac{\partial u}{\partial x_i} \right] - a(t, x_1, \dots, x_n)u \quad (\text{in } Q); \quad u|_{\Gamma} = 0,$$

where $a_i \geq 0$, $a \geq 0$, is approximated by the self-adjoint difference operator

$$L_{hu} = \sum_{i=1}^n \left[a_i(x_i + \frac{1}{2}h_i)u_{x_i} \right]_{\bar{x}_i} - au \quad (\text{in } Q_h); \quad u|_{\Gamma} = 0,$$

where the notation is the same as in (3), then

$$\Lambda = \max_{Q,t} \left[4 \sum \frac{a_i}{h_i^2} + a \right].$$

All schemes are applicable to the approximate solution of the problem

$$\partial u / \partial t = Lu; \quad u|_{\Gamma} = 0; \quad u|_{t=0} = \varphi(A). \quad (4)$$

The estimates (2), (3) remain valid, and convergence holds in the mean with respect to the spatial coordinates and uniformly for $0 \leq t \leq T$, if $\varphi \in L_2(Q)$, while a_i and a are sufficiently smooth functions of x_1, x_2, \dots, x_n , not depending on t .

Remark 4. Exactly as in papers ^{3,4}, one may approximate the operator

$$Lu = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left[a_{ij}(t, x_1, \dots, x_n) \frac{\partial u}{\partial x_j} \right] - a(t, x_1, \dots, x_n)u \quad (\text{in } Q); \quad u|_{\Gamma} = 0,$$

where $a \geq 0$, $a_{ij} = a_{ji}$,

$$\alpha \sum_{i=1}^n \xi_i^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq \beta \sum_{i=1}^n \xi_i^2,$$

α and β are positive constants, by the self-adjoint difference operator

$$L_h u = \sum_{i,j=1}^n \frac{1}{2} [(a_{ij} u_{x_j})_{\bar{x}i} + (a_{ij} u_{\bar{x}j})_{x_i}] - au \quad (\text{in } Q_h); \quad u|_{\Gamma_h} = 0.$$

In this case

$$\Lambda = 4\beta \sum_i \frac{1}{h_i^2} + \max_{Q,t} a.$$

All the schemes considered are applicable to the approximate solution of problem (4). Inequalities (2), (3) remain valid.

Moscow State University
named after M. V. Lomonosov

Received
3 VI 1957

CITED LITERATURE

- ¹ L. A. Lyusternik, Tr. Mat. Inst. im. V. A. Steklova, **20**, 49 (1947).
- ² L. A. Lyusternik, Uspekhi Mat. Nauk, **9**, 2, 3 (1954).
- ³ V. K. Saul' ev, Computational Mathematics and Computational Technology, **2**, 116 (1955).
- ⁴ D. M. Eidus, DAN, **83**, No. 2, 191 (1952).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.