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Abstract

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MATRIX ELEMENTS OF IRREDUCIBLE UNITARY REPRESENTATIONS OF THE GROUP OF EUCLIDEAN MOTIONS OF THREE-DIMENSIONAL SPACE AND THEIR PROPERTIES

(Presented by Academician A. N. Kolmogorov on 25 IX 1956)

The matrix elements of irreducible unitary representations of the group $M(3, R)$ of Euclidean motions of three-dimensional space were first computed by A. M. Rodov by means of the infinitesimal method*. In the present note these matrix elements are computed by an integral method, and some properties of these matrix elements are derived.

The irreducible unitary representations of the group $M(3, R)$ are specified by two parameters: an integer m and a positive number ρ. The representation corresponding to the values of the parameters m and ρ is constructed in the space H_m of functions defined on the group $SO(3)$ of rotations of three-dimensional space, having an integrable square with respect to the invariant measure $d\mu(\omega)$ in $SO(3)$ and satisfying the functional equation

$$f(\gamma\omega) = e^{-im\alpha} f(\omega), \tag{1}$$

where γ is a rotation through an angle α about the axis Oz. To a motion g, consisting of a parallel translation by the vector $\mathbf{h}$ and a subsequent rotation ω_0, there is put in correspondence the operator Q_g, which takes a function $F(\omega)$ from H_m into the function

$$Q_g F(\omega) = e^{-i\rho r \cos\alpha_\omega} F(\omega\omega_0). \tag{2}$$

Here r denotes the length of the vector $\mathbf{h}$, and α_ω denotes the angle between the axis Oz and the vector $\mathbf{h}_\omega$ obtained from the vector $\mathbf{h}$ under the rotation ω.

Let us choose in the space H_m an orthonormal basis consisting of functions of the form

$$\sqrt{2l+1}\,T^l_{mn}(\omega), \qquad l \geq |m|, \quad -l \leq n \leq l \tag{3}$$

(l is an integer), where $T^l_{mn}(\omega)$ are the matrix elements of the irreducible unitary representations of weight l of the group $SO(3)$ (see ([1]), p. 78, where the formula for $T^l_{mn}(\omega)$ is given). In this basis, the operators corresponding to the rotation ω_0 are given by block-diagonal matrices whose main diagonals contain blocks of the form $\|T^l_{nk}(\omega_0)\|$, $l \geq |m|$. The operators Q_g corresponding to a displacement along the axis Oz by a vector of length r are given by matrices consisting of blocks $\|J^{l_1}_l(g)\|$, $l, l_1 \geq |m|$, moreover

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* Reported by A. M. Rodov at the Third All-Union Mathematical Congress in June 1956. Previously reported in the seminar on functional analysis at Moscow State University.

the elements $q_{ln,l_1n_1}(g)$ of the block $\|J^{ll_1}(g)\|$ are equal to zero for $n \neq n_1$ and are equal to $J_{m,l,l_1,n}(\rho r)$ for $n = n_1$, where

$$J_{m,l,l_1,n}(x) =$$

$$= (-1)^{m-n}\frac{\sqrt{(2l+1)(2l_1+1)}}{2}\int_{-1}^{1} e^{-i\mu x} P^l_{mn}(\mu) P^l_{-m,-n}(\mu)\,d\mu. \tag{4}$$

For the functions $P^l_{mn}(\mu)$ see ([1]), p. 78.

Computing the integral (4), we find

$$J_{m,l,l_1,n}(x) =$$

$$= (-1)^{m-n}\frac{\sqrt{(2l+1)(2l_1+1)}}{2}\sum_{k=0}^{l+l_1}\frac{i^{k-1}\left[A_{m,l,l_1,n,k}e^{ix} - B_{m,l,l_1,n,k}e^{-ix}\right]}{x^{k+1}}, \tag{5}$$

where

$$A_{m,l,l_1,n,k} = \left[P^l_{mn}(\mu)P^l_{-m,-n}(\mu)\right]^{(k)}_{\mu=1}, \tag{6}$$

$$B_{m,l,l_1,n,k} = \left[P^l_{mn}(\mu)P^l_{-m,-n}(\mu)\right]^{(k)}_{\mu=-1}. \tag{7}$$

The computation of the constants $A_{m,l,l_1,n,k}$ and $B_{m,l,l_1,n,k}$ is easy to carry out by applying Leibniz' theorem.

Another way of computing the integral (4) consists in replacing the product $P^{l_1}_{mn}(\mu)P^{l}_{-m,-n}(\mu)$ by a linear combination of expressions $P^{k}_{00}(\mu)$. This method leads to the equality

$$J_{m,l,l_1,n}(x) =$$

$$= (-1)^{m-n}\sqrt{\frac{\pi(2l+1)(2l_1+1)}{2x}}\sum_{k=|l-l_1|}^{l+l_1} C^{k,0}_{-n,n}C^{k,0}_{-m,m}J_{k+1/2}(x)i^k, \qquad (8)$$

where $J_{k+1/2}(x)$ are Bessel functions of half-integer index and $C^{k,i+j}_{ij}$ are Clebsch-Gordan coefficients ([2]). In particular,

$$J_{0,l,0,0}(x) = \sqrt{\frac{\pi(2l+1)}{2x}}J_{l+1/2}(x)i^l. \qquad (8')$$

From formulas (5) and (8′) one easily obtains an expression of Bessel functions of half-integer index in terms of trigonometric functions. Formula (8) was obtained by another method by A. M. Rodov.

The functions $J_{m,l,l_1,n}(x)$ satisfy various relations that generalize the corresponding relations for Bessel functions. We shall indicate here the form of the addition theorem for the functions $J_{m,l,l_1,n}(x)$.

Addition theorem. *If the parameters* r_1, r_2, θ *and* r, θ_1 *are related by the relations*

$$r = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos\theta}, \qquad (9)$$

$$\cos\theta_1 = \frac{r_1\cos\theta + r_2}{r}, \qquad (10)$$

then

$$\sum_{k=-N}^{N} J_{m,l_1,l_2,k}(r)\,P^{l_1}_{mk}(\cos\theta_1)\,P^{l_2}_{kn_2}(\cos(\theta-\theta_1)) =$$

$$= \sum_{l=-\infty}^{\infty} J_{m,l_1,l,n}(r_2)J_{m,l,l_2,n_2}(r_1)P^{l}_{n_1n_2}(\cos\theta), \qquad (11)$$

$$N = \min(l_1, l_2).$$

$$J_{m,l_1,l_2,n}(r) =$$

$$= \sum_{l=-\infty}^{\infty} \sum_{s=-l_1}^{l_1} \sum_{t=-l}^{l} (-1)^{s-t} J_{m,l_1,l,s}(r_2) J_{m,l,l_2,t}(r_1) P^{l_1}_{sn}(\cos\theta_1) \times \tag{12}$$

$$\times P^{l}_{st}(\cos\theta) P^{l_2}_{nt}(\cos(\theta-\theta_1)).$$

From the addition theorem (11) there follow recurrence relations for the functions $J_{m,l_1,l_2,n}(x)$. For example, putting $\theta = 0$ in it, differentiating both sides with respect to r_1, and putting $r_1 = 0,\ r_2 = x$, we find that

$$J'_{m,l_1,l_2,k}(x) = \sum_{l=l_2-1}^{l_2+1} J_{m,l_1,l,k}(x) J'_{m,l,l_2,k}(0). \tag{13}$$

Putting $\theta = \pi/2$, differentiating both sides with respect to r_1, and putting $r_1 = 0,\ r_2 = x$, we find

$$\frac{1}{x} P^{l_2}_{n_1,n_2}(0) J_{m,l_1,l_2,n_1}(x) -$$

$$-\frac{i}{2x}\Big[\sqrt{(l_1+n_2)(l_1-n_2+1)}\, P^{l_1}_{n_1,n_2-1}(0) J_{m,l_1,l_2,n_2-1}(x) +$$

$$+\sqrt{(l_1+n_2+1)(l_1-n_2)}\, P^{l_1}_{n_1,n_2+1}(0) J_{m,l_1,l_2,n_2+1}(x)\Big] =$$

$$= \sum_{l=l_2-1}^{l_2+1} P^{l}_{n_1 n_2}(0) J'_{m,l,l_2,n_2}(0) J_{m,l_1,l,n_1}(x). \tag{14}$$

The functions $J_{m,l_1,l_2,n}(x)$ satisfy a second-order differential equation found by A. M. Rodov.

Let us decompose the representation of the subgroup of two-dimensional motions, induced by the representation (2) of the group $M(3,R)$, into irreducible representations. Then one obtains a formula that is a special case of Sonine' s integral (([3]), p. 406) for $\nu = -1/2$ and integral μ. Sonine' s integral for arbitrary integral and half-integral values of μ and ν is obtained by considering representations of the group of n-dimensional motions. Consideration of the group of n-dimensional motions also makes it possible to give a group-theoretic interpretation of the second Sonine integral (([3]), p. 410), as well as of the Gegenbauer-Sonine integral (([3]), p. 400).

Some relations in the theory of Bessel functions are obtained by changing the basis in the representation space. In this way, for example, Sonine' s expansion ([3], p. 153) can be obtained.

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3. G. N. Watson, *Theory of Bessel Functions*, Part I, 1949.

Note: Figure translations are in progress. See original paper for figures.