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Abstract

Full Text

MATHEMATICS

V. B. LIDSKII

A THEOREM ON THE SPECTRUM OF A PERTURBED DIFFERENTIAL OPERATOR

(Presented by Academician A. N. Kolmogorov on 27 IX 1956)

We consider the differential operator

$$Lu = -\Delta u + p_1(x)u, \quad (1)$$

where Δ is the Laplace operator, $x(x_1, x_2, \dots, x_n)$ is a point of the n -dimensional real Euclidean space \mathcal{E}_n , and $p_1(x)$ is a continuous complex function.

It is assumed that the domain of definition of the operator L is some linear manifold $D(L)$, dense in the Hilbert space of complex functions whose squares are integrable over all of \mathcal{E}_n . The Hilbert space is denoted, as usual, by L_2 . It is further assumed that the operator L is closed and has a resolvent R_λ with region of regularity σ . The latter means that for $\lambda \in \sigma$ the equation

$$(L - \lambda E)u = f$$

has, for every $f \in L_2$, a unique solution belonging to $D(L)$, and moreover the linear operator $(L - \lambda E)^{-1} \equiv R_\lambda$ is bounded. Along with the operator (1), we consider the perturbed differential operator

$$\tilde{L}u = -\Delta u + (p_1(x) + p_2(x))u, \quad (2)$$

where $p_2(x)$ is a continuous function satisfying the condition:

$$\lim_{|x| \rightarrow \infty} p_2(x) = 0 \quad \left(|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \right). \quad (3)$$

The perturbed operator \tilde{L} is defined on the same linear manifold $D(L)$ as the unperturbed operator.

The main result of the present work is the following theorem.

Theorem 1. *The perturbed operator*

$$\tilde{L}u = -\Delta u + (p_1(x) + p_2(x))u$$

possesses a resolvent \tilde{R}_λ , which is regular everywhere in the region of regularity σ of the resolvent of the operator L , with the exception of no more than a countable number of points having no limit points inside σ . These points are poles of \tilde{R}_λ .

The theorem just formulated is a generalization of a theorem of I. M. Gel' fand⁽¹⁾, in which the function $p_1(x)$ is assumed real, the operator L self-adjoint, and the perturbing function $p_2(x)$ identically equal to zero outside a finite domain.

For the proof we, as in⁽¹⁾, consider the equation

$$-\Delta u + (p_1(x) + p_2(x))u - \lambda u = f, \quad (4)$$

where $\lambda \in \sigma$, $f \in L_2$, and the solution u is sought among the functions of the linear manifold $D(L)$. Make the substitution

$$u = Rv, \quad (5)$$

where $v \in L_2$. Substituting the expression for u from formula (5) into (4), we obtain, with respect to v , the equation

$$(E + p_2R_\lambda)v = f. \quad (6)$$

Suppose now that the operator $E + p_2R_\lambda$ has a bounded inverse. Then $v = (E + p_2R_\lambda)^{-1}f$ and, by virtue of (5), $u = R_\lambda(E + p_2R_\lambda)^{-1}f$. In this case it is obvious that the solution thus found belongs to $D(L)$, is unique, and the operator $R_\lambda(E + p_2R_\lambda)^{-1}$ is bounded.

Thus, in this case

$$\tilde{R}_\lambda = R_\lambda(E + p_2R_\lambda)^{-1}. \quad (7)$$

The theorem will be proved if we show that the operator p_2R_λ is completely continuous (cf. ⁽¹⁾). The point is that, for operators of the form $E + A(\lambda)$, where $A(\lambda)$ is a completely continuous operator depending analytically on λ , the Fredholm theorems are valid in the same full scope as for operators of the form $E + \lambda P$ with completely continuous P^* . Since the operator p_2R_λ , evidently, is analytic for $\lambda \in \sigma$, the validity of the theorem will be established directly from formula (7).

To prove the complete continuity of the operator p_2R_λ , we shall need one proposition, which we state separately.

Lemma. Denote by (\mathfrak{R}) the ball in the n -dimensional Euclidean space \mathcal{E}_n with center at the origin and radius R . Consider on (\mathfrak{R}) the family of functions M which coincide on this ball with the functions

$$u = R_\lambda f, \quad (8)$$

where

$$\|f\| \leq 1, \quad (9)$$

and R_λ is the resolvent of the operator (1).

The family M is compact in the sense of convergence in the mean.

Proof. From formula (8) it follows that

$$\int_{E_n} |u|^2 dx = \|u\|^2 \leq \|R_\lambda\|^2, \quad (10)$$

whence, a fortiori,

$$\int_{|x| \leq R} |u|^2 dx \leq \|R_\lambda\|^2. \quad (11)$$

We shall now prove, in addition, that

$$\int_{|x| \leq R} |\nabla u|^2 dx \leq C \left(|\nabla u|^2 = \sum_s \frac{\partial u}{\partial x_s} \frac{\partial \bar{u}}{\partial x_s} \right). \quad (12)$$

Thereby the compactness of the family M will be established (see, for example, (2), p. 191).

To prove inequality (12), multiply the relation $-\Delta u + (p_1(x) - \lambda)u = f$, which is equivalent to (8), by the function $\bar{u}(1 - r^2/4R^2)^2$ and integrate both sides of the equality over the ball $|x| \leq 2R$. The integral

$$-\int_{|x| \leq 2R} \Delta u \bar{u} (1 - r^2/4R^2)^2 dx$$

* In paper (1) it is indicated that this remark belongs to M. V. Keldysh.

integrate by parts, putting in the formula

$$\Delta g \cdot h = \operatorname{div}(\nabla g \cdot h) - \nabla g \nabla h,$$

$$g = u, \quad h = \bar{u} \left(1 - \frac{r^2}{4R^2}\right)^2.$$

In this case the surface integral vanishes, and we arrive at the equality

$$\begin{aligned} & \int_{|x| \leq 2R} \nabla u \nabla \bar{u} \left(1 - \frac{r^2}{4R^2}\right)^2 dx - \int_{|x| \leq 2R} \nabla u \nabla r \bar{u} \left(1 - \frac{r^2}{4R^2}\right) \frac{r}{R^2} dx + \\ & + \int_{|x| \leq 2R} (p - \lambda) |u|^2 \left(1 - \frac{r^2}{4R^2}\right)^2 dx = \int_{|x| \leq 2R} f \bar{u} \left(1 - \frac{r^2}{4R^2}\right)^2 dx, \end{aligned} \quad (13)$$

from which we obtain a uniform estimate for the first integral on the left. For this purpose denote the real part of the second of the integrals on the left by I and estimate it. In view of the fact that

$$\nabla u \nabla r = \frac{\partial u}{\partial r}, \quad \frac{\partial u}{\partial r} \bar{u} + \frac{\partial \bar{u}}{\partial r} u = \frac{\partial}{\partial r} |u|^2,$$

we obtain

$$I = \operatorname{Re} \left\{ - \int_{|x| \leq 2R} \nabla u \nabla r \bar{u} \left(1 - \frac{r^2}{4R^2}\right) \frac{r}{R^2} dx \right\} = - \int_{|x| \leq 2R} \frac{r}{R^2} \left(1 - \frac{r^2}{4R^2}\right) \frac{\partial}{\partial r} |u|^2 dx.$$

In the last integral we pass to spherical coordinates and integrate with respect to r from 0 to $2R$ by parts:

$$\begin{aligned} I &= - \int_{|x|=1} \left(\int_0^{2R} \frac{r^n}{R^2} \left(1 - \frac{r^2}{4R^2}\right) \frac{\partial}{\partial r} |u|^2 dr \right) d\omega = \\ &= - \int_{|x|=1} \left(\int_0^{2R} \frac{|u|^2}{R^2} \left[n - (n+2) \frac{r^2}{4R^2} \right] r^{n-1} dr \right) d\omega. \end{aligned}$$

Here it is essential that the second term in the integration by parts is equal to zero.

Thus,

$$I = - \frac{1}{R^2} \int_{|x| \leq 2R} |u|^2 \left(n - (n+2) \frac{r^2}{4R^2} \right) dx,$$

and, by virtue of inequality (10), it follows from this that

$$|I| \leq C_1. \quad (14)$$

We also note that, in view of the boundedness of the function $p(x)$ for $|x| \leq 2R$ and relations (9) and (10), the inequalities

$$\left| \int_{|x| \leq 2R} (p - \lambda) \left(1 - \frac{r^2}{4R^2}\right)^2 |u|^2 dx \right| \leq C_2, \quad \left| \int_{|x| \leq 2R} f\bar{u} \left(1 - \frac{r^2}{4R^2}\right)^2 dx \right| \leq C_3. \quad (15)$$

hold. Separating now the real parts in equality (13), we obtain, by virtue of (14) and (15),

$$\int_{|x| \leq 2R} \nabla u \nabla \bar{u} \left(1 - \frac{r^2}{4R^2}\right)^2 dx \leq C_4. \quad (16)$$

Let us note, however, that

$$\int_{|x| \leq R} \nabla u \nabla \bar{u} dx \leq \frac{16}{9} \int_{|x| \leq 2R} \nabla u \nabla \bar{u} \left(1 - \frac{r^2}{4R^2}\right)^2 dx, \quad (17)$$

for on the ball $|x| \leq R$ the obvious inequality $1 - r^2/4R^2 \geq 3/4$ holds. Inequalities (17) and (16) prove the validity of formula (12). Thus the lemma is proved.

Using the lemma just proved, it is already not difficult to show that the operator $p_2 R_\lambda$ is completely continuous. Indeed, let us show that the family of functions

$$p_2 R_\lambda f, \quad (18)$$

where $\|f\| \leq 1$, is compact in L_2 . Put $R_\lambda f = u$. Let $\varepsilon > 0$ be given. Choose R so large that for $|x| \geq R$ one has $|p_2(x)| \leq \sqrt{\varepsilon/2} |R_\lambda|^{-1}$. By virtue of formula (10) we then obtain

$$\int_{|x| \geq R} |p_2|^2 |u|^2 dx \leq \frac{\varepsilon}{2}. \quad (19)$$

Since on the ball $|x| \leq R$ the family of functions $u = R_\lambda f$ is compact, and the function $p_2(x)$ is bounded, the family of functions $p_2 R_\lambda f$ on the ball under consideration is also compact. Choose for this family a finite $\varepsilon/2$ -net on the ball and extend each function of the net by zero to all of \mathcal{E}^n . In view of estimate (19), in this way we obtain a finite ε -net for the family $p_2 R_\lambda f$. Since ε is arbitrary,

this family is compact in L_2 . This proves the complete continuity of the operator p_2R_λ , and with it the theorem as well.

Let us note a corollary following from the theorem proved.

The spectra of the operators L and \tilde{L} differ from one another by no more than a countable set of points belonging to the discrete part of the spectrum.

Indeed, if λ_0 is not a point of the spectrum of the operator L , then it either does not belong to the spectrum of the perturbed operator, or is a point of the discrete part of its spectrum. Considering, on the other hand, the operator L as the result of perturbing the operator \tilde{L} by the function $p_2(x)$, we can draw the converse conclusion.

On the basis of the corollary formulated, one may, for example, assert that the spectrum of the equation

$$-\Delta u + ip(x)u = \lambda u,$$

where $p(x)$ is any continuous complex function tending to zero as $|x| \rightarrow 0$, consists of all points of the positive half-plane (λ) and, possibly, of a countable set forming the discrete part of the spectrum.

Moscow Institute of Physics and Technology

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