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Mathematics

R. E. Vinograd

1957

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Abstract

Full Text

Mathematics

R. E. Vinograd

ESTIMATE OF THE JUMP OF THE LARGEST CHARACTERISTIC EXPONENT UNDER SMALL PERTURBATIONS

(Presented by Academician I. G. Petrovskii, December 6, 1956)

Let two systems of differential equations of order n be given in matrix-vector form: the unperturbed one

$$\frac{dx}{dt} = A(t)x \quad (1)$$

and the perturbed one

$$\frac{dx}{dt} = A(t)x + f(t, x). \quad (2)$$

Here $A(t)$ is a continuous or piecewise-continuous bounded matrix for $0 \leq t < \infty$; $|A(t)| \leq M$; x and f are vectors, with $f(t, 0) \equiv 0$; f is continuous in t, x and satisfies, with respect to x , a Lipschitz condition with small constant δ , independent of t , or the condition

$$|f(t, x)| \leq \delta|x|. \quad (3)$$

The class of vector-additions $f(t, x)$ possessing Lipschitz constant $\leq \delta$ will be denoted by L_δ .

Let λ_0 and λ_f be the largest characteristic exponents of the solutions of systems (1) and (2), respectively*. It is known^(1,2) that λ_f , while being $\geq \lambda_0$, does not always tend to λ_0 as the additions f decrease, i.e., if one sets

$$\Lambda = \lim_{\delta \rightarrow 0} \sup_{f \in L_\delta} \lambda_f, \quad (4)$$

then the case $\Lambda > \lambda_0$ is possible (for exceptions see⁽³⁻⁵⁾).

Thus, in passing from (1) to (2), the largest exponent may undergo a finite jump $\Lambda - \lambda_0 > 0$ (up to a quantity tending to zero together with δ). In this paper an upper estimate for this jump—or, equivalently, for Λ —is constructed from system (1), and in a certain sense cannot be improved.

1. Let $X(t)$ be the matrix of a fundamental system of solutions of (1). Consider a bounded function $F(t)$ such that, for all t and $\tau < t$, the estimate

$$|X(t)X^{-1}(\tau)| \leq C \exp \left[\int_{\tau}^t F(\xi) d\xi \right] \quad (5)$$

holds, where C depends on the choice of $F(t)$, but not on t and τ . The class of functions $F(t)$ is nonempty; it includes all sufficiently large constants ($\geq M \geq |A(t)|$).

* The characteristic exponent of a vector $x(t)$ is defined as

$$\lambda = \overline{\lim}_{t \rightarrow \infty} t^{-1} \ln |x(t)|;$$

λ_f is understood as the supremum of λ over all solutions $x(t)$ of system (2).

Set

$$\omega_F = \overline{\lim}_{t \rightarrow \infty} t^{-1} \int_0^t F(\xi) d\xi, \quad \Omega = \inf \omega_F, \quad (6)$$

where the infimum is taken over all functions of the indicated class. The number Ω so defined does not depend on the choice of the fundamental matrix $X(t)$ and is not changed under Lyapunov transformations ($x = S(t)y$ with bounded matrices S and S^{-1}) of system (1). Therefore one may transform systems (1) and (2) by Perron's method^(6,7), i.e., regard the matrix $A(t)$ as triangular.

Lemma. The number Ω is determined only by the elements of the main diagonal of the matrix $A(t)$, if the latter is triangular.

Without dwelling on the proof, we note the following construction. Let $a_{11}(t), a_{22}(t), \dots, a_{nn}(t)$ be the elements of the main diagonal of $A(t)$. Divide the semiaxis $0 \leq t < \infty$ into equal intervals Δ_i of length T , and consider the function $\alpha_T(t)$ which on each Δ_i coincides with that one of the functions $a_{kk}(t)$ which has the largest integral over Δ_i . Put

$$\omega_T^* = \overline{\lim}_{t \rightarrow \infty} t^{-1} \int_0^t \alpha_T(\xi) d\xi, \quad \Omega^* = \inf_{0 < T < \infty} \omega_T^*.$$

Then one can prove that Ω^* coincides with Ω .

Theorem 1. For any $\varepsilon > 0$ there exists a sufficiently small $\delta > 0$ such that every solution $x(t)$ of system (2) with $f \in L_\delta$ admits the estimate

$$|x(t)| \leq |x(0)| B_\varepsilon e^{(\Omega + \varepsilon)t}, \quad (7)$$

where B_ε depends only on ε .

Proof. Take the matrix $X(t)$ of solutions of (1) with the initial condition $X(0) = E$, and choose the function $F(t) = F_\varepsilon(t)$ from (5) so that $\omega_F < \Omega + \sigma$, where $\sigma = \varepsilon/3$; denote the corresponding constant by $C = C_\varepsilon$. Replacing (2) by the integral equation

$$x(t) = X(t)x_0 + \int_0^t X(t)X^{-1}(\tau)f(\tau, x(\tau)) d\tau, \quad x_0 = x(0),$$

we shall solve it by Picard' s method (whose convergence is known).

Let the first approximation be $x_1(t) = X(t)x_0$. Then from (5) we obtain

$$|x_1(t)| \leq |x_0| C \exp \left[\int_0^t F d\xi \right]. \quad (8)$$

Suppose that, for $r = 1, 2, \dots, k$, the estimate

$$|x_r(t)| \leq |x_0| C \exp \left[\int_0^t (F + \sigma) d\xi \right] \quad (9)$$

has been established.

Then for the $(k + 1)$ -st approximation, using (8), (5), (9), and the Lipschitz condition for f , we find

$$\begin{aligned} |x_{k+1}(t)| &\leq |X(t)x_0| + \int_0^t |X(t)X^{-1}(\tau)| \delta |x_k(\tau)| d\tau \leq \\ &\leq |x_0| C \exp \left[\int_0^t F d\xi \right] + \int_0^t C \exp \left[\int_\tau^t F d\xi \right] \delta |x_0| C \exp \left[\int_0^\tau (F + \sigma) d\xi \right] d\tau = \\ &= |x_0| C \exp \left[\int_0^t F d\xi \right] \left[1 + C\delta \int_0^t e^{\sigma\tau} d\tau \right] = |x_0| C \exp \left[\int_0^t F d\xi \right] \left[1 + \frac{C\delta}{\sigma} (e^{\sigma t} - 1) \right]. \end{aligned}$$

We see that if δ is so small that $C\delta/\sigma < 1$, then $x_{k+1}(t)$ also satisfies inequality (9). Therefore it is also valid for the desired solution $x(t) = \lim_{k \rightarrow \infty} x_k(t)$:

$$|x(t)| \leq |x_0| C \exp \left[\int_0^t (F + \sigma) d\xi \right]. \quad (10)$$

Taking into account that $\omega_F < \Omega + \sigma$, we have

$$\int_0^t F d\xi < D_\sigma + (\Omega + 2\sigma)t,$$

where D_σ depends only on $\sigma = \varepsilon/3$, and then (10) turns into (7), where $B = Ce^D$, and the theorem is proved.

From this theorem and definitions (4) and (6) it follows:

Corollary. Ω serves as an upper estimate for Λ , i.e. always

$$\Lambda \leq \Omega. \tag{11}$$

2. Let us pass to questions of the unimprovability of this estimate. Considering the matrix $A(t)$ in (1), (2) triangular, we divide the set of such matrices, and thereby also the set of systems (1), into classes, assigning to one class matrices with the same main diagonal. (Such a division is ambiguous because of the nonuniqueness of the Perron transformation, which in the present case is immaterial.) According to the lemma, the number Ω will be constant in each class.

Theorem 2. *In each class there is an infinite set of systems (1) for which estimate (11) becomes an exact equality.*

Omitting the proof, we point out that the desired set of matrices $A(t)$ (systems (11)) contains, in particular, all those matrices for which the off-diagonal elements are ≥ 0 . Thus, in each class this set contains domains, if one introduces the topology as uniform closeness of the off-diagonal elements on the half-axis $0 \leq t < \infty$.

The shortcomings of such a topologization are obvious; moreover, the possibility remains unclear of improving estimate (11) if system (1) does not belong to the mentioned set. However, in these cases there exists another point of view, clarifying the impossibility of improving estimate (11) in a different sense. As shown in ⁽⁸⁾, in the nonlinear case $f(t, x)$ knowledge of the number Λ does not yet make it possible to write, for all solutions of system (2), an estimate of the form

$$|x(t)| \leq |x(0)|B_\varepsilon e^{(\Lambda+\varepsilon)t},$$

since the constant B_ε may here turn out to depend not only on ε , but also on the solution $x(t)$. In turn, this means ⁽⁸⁾ that by means of the number Λ one cannot investigate stability in the way this is done by Lyapunov's first method. As for estimate (7), according to Theorem 1 it does not possess such a shortcoming.

Therefore, regardless of whether (11) is an exact equality, one has to use the number Ω , and not Λ , and thereby the need to refine (11) disappears.

In conclusion, we indicate a generalization of Lyapunov' s theorem on stability by the first approximation.

Theorem 3. *If the added terms $f(t, x)$ have order higher than one in x and if $\Omega < 0$, then the trivial solution of system (2) is asymptotically stable.*

Moscow Aviation Technological
Institute

Received
26 III 1956

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Note: Figure translations are in progress. See original paper for figures.

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