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Abstract

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MATHEMATICS

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SUFFICIENT CONDITIONS FOR NONOSCILLATION AND OSCILLATION OF SOLUTIONS OF THE EQUATION $y^{''}+p(x)y=0$

(Presented by Academician I. G. Petrovsky, 8 X 1956)

It is known ([1],[6],[7])* that the solutions of the equation

$$y''+p(x)y=0 \tag{1}$$

are nonoscillatory for $x \geq x_0$, if there exists a continuously differentiable function $\theta(x)$ such that

$$\theta'+\theta^2+p\leq 0,\qquad x\geq x_0. \tag{2}$$

Using this, it is easy to obtain a series of sufficient conditions under which the solutions of equation (1) will be nonoscillatory.

Theorem 1. If for $x \geq x_0$ there exist a positive differentiable function $r(x)$ and a constant $\nu \geq 0$ such that, for some constant C, the inequality

$$\frac{r'}{2}-\frac{r'}{2}\sqrt{\nu}\leq -\int_{x_0}^{x}\left[p-\frac{(1-\nu){r'}^2}{4r^2}\right]r\,dx+C\leq \frac{r'}{2}+\frac{r'}{2}\sqrt{\nu},\qquad x\geq x_0,$$

is satisfied, then the solutions of equation (1) are nonoscillatory.

Under the conditions of the theorem,

$$\theta_1(x)=\frac{1}{r}\left(\int_a^x\left[p-\frac{(1-\nu){r'}^2}{4r^2}\right]r\,dx+C\right)$$

satisfies condition (2). Indeed, if we denote

$$-\int_{x_0}^{x}\left[p-\frac{(1-\nu){r'}^2}{4r^2}\right] r\,dx + C = u,$$

we obtain:

$$\theta_1'+\theta_1^2+p = (ru'+r'u+u^2+pr^2)/r^2 = [-pr^2+\tfrac{1}{4}(1-\nu){r'}^2-r'u+u^2+pr^2]/r^2 = [u-\tfrac{1}{2}(r'-r'\sqrt{\nu})][u-\tfrac{1}{2}(r'+r'\sqrt{\nu})]/$$

Corollary 1. If $\nu = 1$, Petropavlovsky' s theorem ([2]) is obtained.

Corollary 2. If $r = x$, it follows that, for nonoscillation, it is sufficient that, for some $\nu \geq 0$, the inequality

$$0 \leq -\int_c^x xp\,dx - \frac{1-\nu}{4}\ln x \leq \sqrt{\nu}$$

be satisfied.

Corollary 3. If $r = x^\alpha$, $\alpha \geq 0$, then we obtain the sufficient condition for nonoscillation

$$0 \leq -\int_a^x px^\alpha\,dx + C \leq \alpha x^{\alpha-1}$$

for some C and $\alpha \geq 0$.

* In note ([7]) an elementary proof of this fact is given.

Theorem 2. If there exists a differentiable function $r(x)$ such that

$$0 \leqslant -\int_a^x \left(px - \frac{1}{4x}\right) r\,dx + C \leqslant r'x, \qquad a \leqslant x \leqslant b,$$

for at least one constant C, then the solutions of equation (1) are nonoscillatory.

When the conditions of the theorem are fulfilled, it is also easy to find a function $\theta(x)$ satisfying the inequality $\theta' + \theta^2 + p \leqslant 0$. Such a function, for example, is

$$\theta(x) = \frac{1}{x^2}\left[-\int_a^x \left(px - \frac{1}{4x}\right) r\,dx + C\right] + \frac{1}{2x}.$$

Corollary. If we take $r = (\ln x)^\alpha$, $\alpha \geqslant 0$, then we obtain that, for nonoscillation, it is sufficient that

$$0 \leqslant -\int_a^x \left(px - \frac{1}{4x}\right)(\ln x)^\alpha\, dx + C \leqslant \alpha(\ln x)^{\alpha-1}.$$

Theorem 2 gives an admissible measure, for preserving nonoscillation, of the deviation of the function $p(x)$ from $1/4x^2$, while Theorem 1 gives a measure of such a deviation from zero.

Theorem 3. If there exists a differentiable function $r(x) > 0$ such that

$$\lim_{x\to+\infty}\int_a^x \left(pr - \frac{{r'}^2}{4r}\right) dx = +\infty,$$

then every solution of equation (1) has an infinite number of zeros on $[a, \infty)$.

Suppose the contrary. Then there will exist, continuous for $x \geqslant x_0$, a solution of the equation $\theta' + \theta^2 + p = 0$ (3). The function $u = \theta r$ satisfies the equation $ru' - r'u + u^2 + pr^2 = 0$. Hence $u' - {r'}^2/4r + pr \leqslant 0$. Integrating, we obtain $\lim_{x\to+\infty} u(x) = -\infty$, whence it follows that all functions $\theta(x)$ are negative for $x \geqslant x_1$, which is impossible (4).

From this theorem, taking $r = x^\alpha$ $(\alpha < 1)$, one obtains Hille' s result (5): if

$$\int^\infty px^\alpha dx = \infty \quad (\alpha < 1),$$

then all solutions of equation (1) have an infinite number of zeros. Let us note that it can be proved that, if $\int^\infty (px^\alpha)\, dx < \infty$ $(0 \leqslant \alpha < 1)$, then the solutions have few zeros; the series

$$\sum_{n=1}^\infty \frac{1}{(a_{n+1} - a_n)^{1-\alpha}},$$

where a_n is the sequence of zeros of an arbitrary solution, converges.

Theorem 4. If there exists a differentiable function $r(x) > 0$ such that

$$\lim_{x\to+\infty}\int_a^x \left[\left(px - \frac{1}{4x}\right) r - \frac{x{r'}^2}{4r}\right] dx = +\infty,$$

then all solutions of equation (1) have an infinite number of zeros for $a \leqslant x < +\infty$.

The proof is analogous to the proof of Theorem 3; instead of the function θr, one must consider $u = (\theta - 1/2x)rx$.

Corollary. If for the function $r(x)$ we take $(\ln x)^\alpha,\ \alpha < 1$, then we obtain the sufficient condition for oscillation:

$$\int_a^\infty \left(px - \frac{1}{4x}\right)(\ln x)^\alpha dx = +\infty.$$

Theorem 5. If the solutions of the equation $y'' + \varphi(x)y = 0$ have on $[a, \infty)$ an infinite number of zeros, and moreover

$$\left|\int^\infty \varphi\, dx\right| < \infty, \qquad \left|\int^\infty p\, dx\right| < \infty$$

and

$$\int_x^\infty p\, dx \geq \int_x^\infty \varphi\, dx \geq 0,$$

then the solutions of equation (1) also have an infinite number of zeros.

Suppose the contrary. Then the equation $\theta' + \theta^2 + p = 0$ has an indefinitely continuable solution. Making the substitution

$$\psi(x) = \theta - \int_x^\infty p\, dx, \tag{*}$$

we obtain

$$\psi' + \left[\psi + \int_x^\infty p\, dx\right]^2 = 0. \tag{**}$$

Among the indefinitely continuable solutions of this equation there is a positive one; otherwise, since by virtue of (**) ψ is a monotonically nonincreasing function, all $\theta(x)$, by virtue of (*), would be negative beginning with some x, which is impossible ([4]). Denote the positive solution by ψ_1. The function

$$\sigma(x) = \psi_1 + \int_x^\infty \varphi\, dx$$

must satisfy the condition $\sigma' + \sigma^2 + \varphi \leq 0$, but this is impossible, since it was assumed that the equation $y'' + \varphi(x)y = 0$ has oscillating solutions. From this theorem, for example, it follows that if

$$\int_x^\infty p\, dx \geq \frac{1+\varepsilon}{4x}, \qquad \varepsilon > 0,$$

then all solutions of equation (1) are oscillating.

Theorem 6. If for some x_0 and $\alpha < 1$ the inequality

$$\int_{x_0}^x p x^\alpha dx \geq C > -\infty$$

holds, and

$$\int^{\infty} \frac{g_+^2(x)}{x^\alpha}\,dx = \infty, \qquad \int^{\infty} \frac{g_-^2(x)}{x^\alpha}\,dx = \infty,$$

where

$$g(x) = -\int_{x_0}^{x} px^\alpha dx + \left[\frac{\alpha^2}{4(\alpha-1)} - \frac{\alpha}{2}\right] x^{\alpha-1}$$

(g_+ and g_- are, respectively, the positive and negative parts of $g(x)$), then the solutions of equation (1) have an infinite number of zeros.

Suppose that the conditions of the theorem are satisfied, but the solutions of (1) have a finite number of zeros. Then the equation $\theta' + \theta^2 + p = 0$ has an indefinitely continuable solution. The function $u = \theta x^\alpha$ satisfies the equation

$$u' + \frac{(u - \alpha x^{\alpha-1}/2)^2}{x^\alpha} + px^\alpha - \frac{\alpha^2 x^{\alpha-2}}{4} = 0$$

or

$$u(x) - u(x_0) + \int_{x_0}^{x} \frac{(u - \alpha x^{\alpha-1}/2)^2}{x^\alpha}\,dx + \int_{x_0}^{x} px^\alpha dx - \frac{\alpha^2 x^{\alpha-1}}{4(\alpha-1)} + \frac{\alpha^2 x_0^{\alpha-1}}{4(\alpha-1)} = 0.$$

Among its solutions extendable without bound there will be one such $u_1(x)$ that

$$\int^{\infty} \frac{(u_1 - \alpha x^{\alpha-1}/2)^2}{x^\alpha}\,dx < \infty,$$

otherwise all $u(x) \to -\infty$ and $\theta(x)$ would be < 0 for large x, and this is impossible by (4).

We have

$$u_1(x) = \lambda + \int_{x}^{\infty} \frac{(u_1 - \alpha x^{\alpha-1}/2)^2}{x^\alpha}\,dx + \frac{\alpha^2 x^{\alpha-1}}{4(\alpha-1)} \int_{x_0}^{x} -px^\alpha\,dx,$$

where

$$\lambda = u(x_0) - \int_{x_0}^{\infty} \frac{(u_1 - \alpha x^{\alpha-1}/2)^2}{x^\alpha}\,dx - \frac{\alpha^2 x_0^{\alpha-1}}{4(\alpha-1)}.$$

Suppose $\lambda \geq 0$. Then

$$\int_{x_0}^{x} \frac{(u_1 - \alpha x^{\alpha-1}/2)^2}{x^\alpha}\,dx = \int_{x_0}^{x} \frac{1}{x^\alpha}\left[\lambda + g(x) + \int_{x}^{\infty} \frac{(u_1 - \alpha x^{\alpha-1}/2)^2}{x^\alpha}\,dx\right]^2 dx \geq \int_{x_0}^{x} \frac{g_+^2}{x^\alpha}\,dx \to +\infty.$$

If, however, $\lambda < 0$, then

$$\int_{x_0}^{x} \frac{(u_1 - \alpha x^{\alpha-1}/2)^2}{x^\alpha}\, dx \geq \int_{x_0}^{x} \frac{[|\lambda|/2 + g_-(x)]^2}{x^\alpha}\, dx \geq \int_{x_0}^{x} \frac{g_-^2}{x^\alpha}\, dx \to +\infty,$$

but

$$\int_{x_0}^{\infty} \frac{(u_1 - \alpha x^{\alpha-1}/2)^2}{x^\alpha}\, dx < \infty.$$

The contradiction obtained proves the theorem.

The conditions of the theorem, for example, will be satisfied if

$$\int_{x_0}^{x} px\, dx$$

is a periodic function. We note that the requirement of divergence of

$$\int^{\infty} \frac{g_+^2}{x^\alpha}\, dx$$

and

$$\int^{\infty} \frac{g_-^2}{x^\alpha}\, dx$$

cannot be omitted.

Theorem 7. *If the solutions of the equations*

$$y'' + f_i(x) y = 0 \qquad (i = 1, 2, \dots, n)$$

are nonoscillatory, then the solutions of the equation

$$y'' + \left(\sum_{i=1}^{n} c_i f_i \right) y = 0$$

are also nonoscillatory for $c_i \geq 0$, $\sum_{i=1}^{n} c_i \leq 1$.

Indeed, from the nonoscillation of the solutions of these equations it follows that for each i there exist θ_i such that

$$\theta_i' + \theta_i^2 + f_i \leq 0.$$

Then

$$\theta = \sum_{i=1}^{n} c_i \theta_i$$

satisfies the condition

$$\theta' + \theta^2 + \sum_{i=1}^{n} c_i f_i \leq 0,$$

whence nonoscillation follows. For the case $n = 2$, see ([2]).

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Note: Figure translations are in progress. See original paper for figures.