



Soviet-era science, translated into English

MATHEMATICS

1957

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-195701.16614>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

N. A. LEBEDEV

ON THE RANGE OF VALUES OF A FUNCTIONAL IN THE PROBLEM OF NON-OVERLAPPING DOMAINS

(Presented by Academician V. I. Smirnov on 18 III 1957)

Let B and B^* be two arbitrary simply connected domains lying in the w -plane, having no common points, and such that the domain B contains the point $w = 0$, while the domain B^* contains the point $w = \infty$. Let $w = f(z)$, $f(0) = 0$, be a function regular and univalent in the disk $|z| < 1$ and mapping the disk $|z| < 1$ onto B , and let $w = F(\zeta)$, $F(\infty) = \infty$, be a function regular and univalent in $1 < |\zeta| < +\infty$ and mapping $|\zeta| > 1$ onto B^* . We shall call the set of pairs of functions $(f(z), F(\zeta))$ of the indicated type the class \mathfrak{M} .

In this paper we find the domain E of values of the quantity $\xi = f(z_0)/F(\zeta_0)$, for fixed z_0 and ζ_0 , respectively from $0 < |z_0| < 1$ and $1 < |\zeta_0| < \infty$, in the class \mathfrak{M} (i.e., the set of all those values of the quantity ξ which it assumes when the pair of functions $(f(z), F(\zeta))$ ranges over the whole class \mathfrak{M}).

If $(f(z), F(\zeta)) \in \mathfrak{M}$ and φ and ψ are real numbers, then $(f(ze^{i\varphi}), F(\zeta e^{i\psi})) \in \mathfrak{M}$. Hence we conclude that the domain E of values of the quantity

$$\xi = f(z_0)/F(\zeta_0)$$

coincides with the domain of values of the quantity $\xi = f(r)/F(\rho)$, $r = |z_0|$, $\rho = |\zeta_0|$. Therefore in what follows we put $z_0 = r$, $\zeta_0 = \rho$.

It is easy to prove that the domain E of values of the quantity ξ is a closed connected bounded set with the point $\xi = 0$ removed. The determination of the domain E reduces ⁽¹⁾ to finding the minimum J_0 of the expression

$$J = \left| \frac{f(r)}{F(\rho)} - a \right|$$

in the class \mathfrak{M} , for an arbitrary point $a \in \overline{E}$.

Let $(f(z), F(\zeta))$ be a pair of functions from \mathfrak{M} for which this minimum J_0 of the quantity J is attained. We shall call the functions of such a pair extremal. Relying on the variational method of G. M. Goluzin ⁽²⁾, we obtain: if $(f(z), F(\zeta)) \in \mathfrak{M}$, then, for $|h|$ sufficiently small, the class \mathfrak{M} also contains:

1) the pair of functions

$$f_*(z) = f(z) + h \frac{f(z)}{f(z) - w_0}, \quad F_*(\zeta) = F(\zeta) + h \frac{F(\zeta)}{F(\zeta) - w_0}, \quad (1)$$

if w_0 is an exterior point simultaneously for the domains B and B^* ;

2) the pair of functions

$$\begin{aligned} f_*(z) &= f(z) + h \frac{f(z)}{f(z) - f(z')} - h \left(\frac{f(z)}{z' f'(z')^2} \right) \frac{z f'(z)}{z - z'} \\ &\quad + \bar{h} \frac{\overline{f(z')}}{z' f'(z')^2} \frac{z^2 f'(z)}{1 - z' z} + O(h^2), \\ F_*(\zeta) &= F(\zeta) + h \frac{F(\zeta)}{F(\zeta) - f(z')}, \end{aligned} \quad (2)$$

for $|z'| < 1$;

3) the pair of functions

$$f_*(z) = f(z) + \frac{f(z)}{f(z) - F(\zeta')},$$

$$F_*(\zeta) = F(\zeta) + h \frac{F(\zeta)}{F(\zeta) - F(\zeta')} - h \left(\frac{F(\zeta')}{\zeta'^2 F'(\zeta')^2} \right) \frac{\zeta^2 F'(\zeta)}{\zeta - \zeta'} + \bar{h} \left(\frac{F(\zeta')}{\zeta'^2 F'(\zeta')^2} \right) \frac{\zeta F'(\zeta)}{1 - \bar{\zeta}' \zeta} + O(h^2) \quad (3)$$

for $|\zeta'| > 1$.

Let $(f(z), F(\zeta))$ be a pair of extremal functions from \mathfrak{M} , corresponding to some point $a \in \bar{E}$, and let $(f_*(z), F_*(\zeta))$ be the varied functions obtained by one of the pairs of formulas (1), (2), or (3). Then

$$J_* = \left| \frac{f_*(r)}{F_*(\rho)} - a \right| \geq \left| \frac{f(r)}{F(\rho)} - a \right| = J_0.$$

Relying on this inequality and formulas (1), in the usual way ⁽²⁾ we prove that the domains B and B^* in this case cover the whole w -plane, in the sense that there are no points external simultaneously to both domains B and B^* . If, however, we use formulas (2) and (3), we obtain the following two differential equations for the extremal functions:

$$e^{-i\alpha} \frac{z f'(z)^2}{f(z)} \left(\frac{1}{f(z) - F(\rho)} - \frac{1}{f(z) - f(r)} \right) = \frac{A}{(r - z)(1 - rz)},$$

$$e^{-i\alpha} \frac{\zeta F'(\zeta)^2}{F(\zeta)} \left(\frac{1}{F(\zeta) - F(\rho)} - \frac{1}{F(z) - f(r)} \right) = \frac{B}{(\rho - \zeta)(1 - \rho\zeta)}, \quad (4)$$

where

$$\alpha = \pi - \arg \left\{ \left(\frac{f(r)}{F(\rho)} - a \right) \frac{f(r)}{F(\rho)} \right\}, \quad 0 \leq \alpha < 2\pi;$$

$$A = e^{-i\alpha} \frac{r f'(r)}{f(r)} (1 - r^2) > 0; \quad B = e^{-i\alpha} \frac{\rho F'(\rho)}{F(\rho)} (\rho^2 - 1) > 0.$$

From the fact that the quantity

$$J = \left| \frac{f(re^{i\varphi})}{F(\rho)} - a \right|, \quad -\pi \leq \varphi \leq \pi,$$

attains its minimum at $\varphi = 0$, we conclude that A is a real number. From the fact that the quantity

$$J = \left| \frac{f(\psi(r, t))}{F(\rho)} - a \right|,$$

where $\psi(z, t)$ is the solution of the Loewner equation

$$\frac{\partial \psi}{\partial t} = -\psi \frac{1 + k\psi}{1 - k\psi}, \quad k = \text{const},$$

$$|k| = 1, \quad \psi|_{t=0} = z, \quad t \geq 0,$$

attains its minimum at $t = 0$, we conclude that $A > 0$. Analogously, we prove that $B > 0$.

Analyzing the differential equations (4), we conclude that the domains B and B^* have as a common boundary some closed simple analytic curve.

Extracting the square root from both sides of equations (4) and integrating the first of the resulting equalities with respect to z from 0 to r , and the second with respect to ζ from

ρ to ∞ , we have

$$-e^{-i\alpha/2} K \left(\sqrt{\frac{\xi}{\xi - 1}} \right) = \sqrt{A} K(r) = \sqrt{B} \frac{1}{\rho} K \left(\frac{1}{\rho} \right),$$

$$K(k) = \int_0^{\pi/2} \frac{du}{\sqrt{1 - k^2 \sin^2 u}} \quad (5)$$

(where by the radical $\sqrt{1 - k^2 \sin^2 u}$ we mean that branch which becomes unity at $u = 0$).

Mark in the w -plane the point $F(1)$. Let this point coincide with the point $f(e^{i\varphi_0})$, $\pi \leq \varphi_0 < 3\pi$. We shall now integrate the equalities obtained from (4), after taking the square root, with respect to z from 0 to -1 , then along the arc $|z| = 1$ counterclockwise from -1 to $e^{i\varphi}$, and, finally, with respect to ζ from 1 to ρ . Adding the equalities thus obtained, we have

$$\begin{aligned} -e^{-i\alpha/2} K\left(\sqrt{\frac{1}{1-\xi}}\right) &= \sqrt{A} \frac{1}{2} K(\sqrt{1-r^2}) + \\ &+ \sqrt{B} \frac{1}{2\rho} K\left(\sqrt{1-\frac{1}{\rho^2}}\right) + \sqrt{A} K(r)\lambda i, \end{aligned}$$

where $0 \leq \lambda < 2$.

Eliminating from this equality the numbers \sqrt{A} and B , with the aid of equalities (5), we obtain

$$\frac{K\left(\sqrt{\frac{1}{1-\xi}}\right)}{K\left(\sqrt{\frac{\xi}{\xi-1}}\right)} = \frac{1}{2} \left\{ \frac{K(\sqrt{1-r^2})}{K(r)} + \frac{K\left(\sqrt{1-\frac{1}{\rho^2}}\right)}{K\left(\frac{1}{\rho}\right)} \right\} + i\lambda = \mu. \quad (6)$$

Now, instead of the parameter α , one should consider the parameter λ .

From (6), relying on the known formulas ⁽³⁾ (p. 321), we have

$$\xi = \xi(\lambda) = -4^2 q \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 - q^{2n-1}} \right)^8, \quad q = e^{-\pi\mu}. \quad (7)$$

Hence we conclude that $\xi(\lambda)$ ($0 \leq \lambda < 2$) is a single-valued function. As λ varies in the interval $[0; 2)$, the point $\xi = \xi(\lambda)$ describes the boundary of the domain E of values of the quantity $\xi = f(r)/F(\rho)$ in the class \mathfrak{M} (the extremal functions corresponding to a given λ can also be found). The domain E is symmetric with respect to the real axis.

Consequences.

1. From formula (7) we have*

$$\left| \frac{f(r)}{F(\rho)} \right| \leq -\xi(0) \leq \frac{r}{\sqrt{(1-r^2)(\rho^2-1)}}.$$

In the first of the inequalities the equality sign is realized for all r , $0 < r < 1$, and ρ , $1 < \rho < +\infty$; in the second—only for $\rho = 1/r$.

2. In the class \mathfrak{M} we have the estimate ((³), p. 321)

$$\left| \ln \left(1 - \frac{f(r)}{F(\rho)} \right) \right| \leq \ln(1 - \xi(0)) \leq \ln \left(1 + \frac{r}{\sqrt{(1-r^2)(\rho^2-1)}} \right)^{**}.$$

* The estimate $\left| \frac{f(r)}{F(\rho)} \right| \leq \frac{r}{\sqrt{(1-r^2)(\rho^2-1)}}$ was obtained by another method by Yu. E. Alenitsyn (⁴).

** By $\ln(1-w)$ is meant that branch which becomes zero at $w = 0$.

Here, in the first of the inequalities the equality sign is attained for all r and ρ , while in the second—only when $\rho = 1/r$.

3. Relying on formulas (7), we obtain: the range of values of the quantity $f'(0)/F(\rho)$ (or $f(r)/F'(\infty)$) in the class \mathfrak{M} is the disk

$$\left| \frac{f'(0)}{F(\rho)} \right| \leq 4 \exp \left[-\frac{\pi}{2} \frac{K(\sqrt{1-1/\rho})}{K(1/\rho)} \right]$$

$$\left(\left| \frac{f(r)}{F'(\infty)} \right| \leq 4 \exp \left[-\frac{\pi}{2} \frac{K(\sqrt{1-r^2})}{K(r)} \right] \right) \quad (8)$$

with the center removed. Hence we easily obtain the known estimate in the class \mathfrak{M} :

$$\left| \frac{f'(0)}{F'(\infty)} \right| \leq 1.$$

4. The range of values of the system $\left(|f(r)|, \frac{1}{|F(\rho)|} \right)$ in the class \mathfrak{M} is determined by the inequalities

$$0 < |f(r)|, \frac{1}{|F(\rho)|} \leq -\xi(0).$$

5. Let C be the class of functions $f(z)$, $f(0) = 0$, regular and univalent in the disk $|z| < 1$ and satisfying there the condition

$$f(z_1)f(z_2) \neq 1$$

(the Bieberbach-Eilenberg class), and let C' be the subclass of univalent functions from C . It follows from (7) that the range of values of the quantity $f(z)$ in the class C' is the closed domain with the point $\xi = 0$ removed, bounded by the curve

$$\xi = \xi_1(\lambda) = 4iq^{1/2} \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1-q^{2n-1}} \right)^4,$$

$$q = e^{-\pi\mu_1}, \quad \mu_1 = \frac{K(\sqrt{1-|z|^2})}{K(|z|)} + i\lambda, \quad 0 \leq \lambda < 4.$$

Received
15 III 1957

CITED LITERATURE

- ¹ N. A. Lebedev, *Vestn. Leningradsk. univ.*, No. 8, 29 (1955).
- ² G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Moscow-Leningrad, 1952.
- ³ I. M. Ryzhik, I. S. Gradshtein, *Tables of Integrals, Sums, Series, and Products*, Moscow-Leningrad, 1951.
- ⁴ Yu. E. Alyoshin, *DAN*, 115, No. 6 (1957).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.