ON THE STRUCTURE OF GROUPS WITH FINITE CLASSES OF CONJUGATE ELEMENTS

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

S. N. CHERNIKOV

ON THE STRUCTURE OF GROUPS WITH FINITE CLASSES OF CONJUGATE ELEMENTS

(Presented by Academician A. N. Kolmogorov, 23 I 1957)

In the present note a mixture is established of mixed groups all of whose classes of conjugate elements are finite, with locally normal groups.

1. Lemma 1. *If the factor group of a torsion-free group $\mathfrak{G}$ by its center $\mathfrak{Z}$ is locally finite, then the group $\mathfrak{G}$ is abelian.*

Proof. If, among the groups satisfying the conditions of this proposition, there exist nonabelian groups with finite factor group by the center, then among the latter choose a group $\mathfrak{G}^*$ with minimal order of the factor group $\mathfrak{G}^*/\mathfrak{Z}$ by its center $\mathfrak{Z}$. Then all proper subgroups of the group $\mathfrak{G}^*$ containing the center $\mathfrak{Z}$ must be abelian, and hence all proper subgroups of the factor group $\mathfrak{G}^*/\mathfrak{Z}$ must be abelian. Hence the solvability of this factor group follows (^{1}).

In view of the solvability of the group $\mathfrak{G}^*/\mathfrak{Z}$, it has a normal divisor $\overline{\mathfrak{G}}$ of some prime index p. Since $\mathfrak{G} \neq \mathfrak{G}^*$, the subgroup $\overline{\mathfrak{G}}$ is abelian. If X is any representative of the coset generating the factor group $\mathfrak{G}^*/\overline{\mathfrak{G}}$, and $\{X\}$ is the cyclic group generated by it, then

$$\mathfrak{G}^* = \{X\}\overline{\mathfrak{G}}, \qquad X^p \subset \overline{\mathfrak{G}}.$$

Since all classes of conjugate elements of the group $\mathfrak{G}^*$, and hence also of the group $\mathfrak{G}^*/\{X^p\}$, are finite, its element $X \cdot \{X^p\}$ of finite order is contained in some finite normal divisor $\mathfrak{N}/\{X^p\}$ (see (^{2}), Dietzmann' s lemma). All proper subgroups of the group $\mathfrak{N}$, obviously, have finite index in it, and therefore, being a torsion-free group, it must be cyclic (^{3}). Thus the group $\{X\}$ is contained in the cyclic normal divisor $\mathfrak{N}$ of the group $\mathfrak{G}^*$. Since all subgroups of a cyclic group are its characteristic subgroups, it follows that the subgroup $\{X\}$ is invariant in the group $\mathfrak{G}^*$. But then the group $\mathfrak{G}^*/\{X^p\}$ is abelian, and hence $\mathfrak{G}^*$ is a nilpotent group of class 2.

By virtue of the servility of the center in an arbitrary nilpotent torsion-free group (^{2}) and the finiteness of the factor group $\mathfrak{G}^*/\mathfrak{Z}$, it follows that $\mathfrak{G}^* = \mathfrak{Z}$. However, this contradicts the choice of the group $\mathfrak{G}^*$ (the group $\mathfrak{G}^*$ is nonabelian). Consequently, among the groups satisfying the conditions of the proposition being proved, there do not exist nonabelian groups with finite factor group by the center.*

Thus, if $\mathfrak{G}$ is an arbitrary nonabelian group satisfying the conditions of the proposition being proved, then the factor group $\mathfrak{G}/\mathfrak{Z}$

---

* This proposition also follows from the results of (^{4}).

is infinite. Since, by virtue of the proposition already proved, all finite subgroups of the factor group $\mathfrak{G}/\mathfrak{Z}$ are abelian, in view of its local finiteness it is abelian. But then the group $\mathfrak{G}$ will be a nilpotent group of class 2. Hence, in view of the servancy of the center of an arbitrary torsion-free nilpotent group, it follows that $\mathfrak{G} = \mathfrak{Z}$. However, this contradicts the infiniteness of the factor group $\mathfrak{G}/\mathfrak{Z}$. Consequently, the group $\mathfrak{G}$ is abelian. The lemma is proved.

2. **Lemma 2.** *If an abelian subgroup $\mathfrak{A}$ of a group $\mathfrak{G}$ with finite classes of conjugate elements has in $\mathfrak{G}$ a centralizer of finite index, then its intersection with the center $\mathfrak{Z}$ of the group $\mathfrak{G}$ has finite index in it.*

Proof. Let $z(\mathfrak{A})$ be the centralizer of the subgroup $\mathfrak{A}$ in $\mathfrak{G}$, and let M be the set consisting of representatives of the distinct cosets $X \cdot z(\mathfrak{A})$, one taken from each class, and let $z(M)$ be its centralizer in $\mathfrak{G}$. From the finiteness of the set M and the finiteness of all classes of conjugate elements of the group $\mathfrak{G}$ it follows that $z(M)$ has finite index in $\mathfrak{G}$. Hence it follows that the intersection $\mathfrak{A} \cap z(M)$ has finite index in $\mathfrak{A}$. Since $\mathfrak{A} \cap z(M) \subset \mathfrak{Z}$, our assertion is proved.

Corollary. *If all classes of conjugate elements of the group $\mathfrak{G}$ are finite, then some power of each element of infinite order of the group $\mathfrak{G}$ is contained in its center.*

Theorem 1. *A group $\mathfrak{G}$ has no infinite classes of conjugate elements if and only if either it is locally normal, or its center contains a torsion-free subgroup $\mathfrak{A}$ such that the factor group $\mathfrak{G}/\mathfrak{A}$ is locally normal.*

Proof. Let $\mathfrak{G}$ be a group with finite classes of conjugate elements. If the group $\mathfrak{G}$ is periodic, then it is locally normal by Dietzmann' s lemma (^{2}). If the group $\mathfrak{G}$ is not periodic, then, in view of the corollary to Lemma 2, the factor group of the group $\mathfrak{G}$ by its center $\mathfrak{Z}$ will be periodic. Hence there follows the existence in the center $\mathfrak{Z}$ of the group $\mathfrak{G}$ of such a torsion-free subgroup $\mathfrak{A}$ which determines a periodic factor group $\mathfrak{G}/\mathfrak{A}$. By Dietzmann' s lemma the latter is locally normal. The necessity of the condition of the theorem is proved.

Passing to the proof of the sufficiency of the condition of the theorem, we shall first show that all elements of finite order of the group $\mathfrak{G}$, which is an extension of the torsion-free subgroup $\mathfrak{A}$ from its center by means of a locally normal group, form a subgroup. Indeed, let X and Y be two arbitrary elements of finite order from the group $\mathfrak{G}$. Since the group $\mathfrak{G}/\mathfrak{A}$ is locally normal and, hence, locally finite, its subgroup

$$\{X\mathfrak{A},\ Y\mathfrak{A}\} = \{X,\ Y\}\mathfrak{A},$$

generated by its elements $X\mathfrak{A}$ and $Y\mathfrak{A}$, is finite. But then, in view of the relation

$$\{X,\ Y\}\mathfrak{A}/\mathfrak{A} \cong \{X,\ Y\}/\mathfrak{A} \cap \{X,\ Y\},$$

the group $\{X,\ Y\}$, generated by the elements X and Y, will be a finite extension of its center and, hence, a group with finite classes of conjugate elements. Relying on Dietzmann' s lemma, from this we obtain without difficulty that the elements of finite order of the group $\{X,\ Y\}$ form a subgroup. Since this subgroup contains the elements X and Y, the group $\{X,\ Y\}$ is periodic. In view of the arbitrariness of the elements X and Y, it follows that the elements of finite order of the group $\mathfrak{G}$ form a subgroup in it. Denote this subgroup by $\mathfrak{R}$. In view of its uniqueness it is invariant in $\mathfrak{G}$.

The factor group $\mathfrak{G}/\mathfrak{R}$ is a torsion-free group. It may be regarded as an extension of the group $\mathfrak{A}/\mathfrak{R}$, contained in its center, by the locally normal (and hence locally finite) group $\mathfrak{G}/\mathfrak{A}$ (the group $\mathfrak{G}/\mathfrak{A}$ is locally normal by virtue of its isomorphism to a factor group of the locally normal group $\mathfrak{G}/\mathfrak{A}$). But then, by Lemma 1, the group $\mathfrak{G}/\mathfrak{R}$ must be abelian. It follows that the commutator of any two elements of the group $\mathfrak{G}$ is contained in $\mathfrak{R}$ and therefore has finite order. Thus, for two arbitrary elements X and Y of the group $\mathfrak{G}$, we obtain the relation

$$Y^{-1}XY = XR, \tag{1}$$

where R is an element of finite order.

Let now Z be some element of the group $\mathfrak{G}$ defining an infinite class of elements conjugate to it, and let

$$Z_1, Z_2, \dots, Z_n, \dots$$

be some sequence of mutually distinct elements of the group $\mathfrak{G}$ conjugate to Z. Using relation (1), write an arbitrary element Z_n of this sequence in the form of a product ZR_n, where R_n is some element of finite order. Since the factor group $\mathfrak{G}/\mathfrak{A}$ is locally normal and the adjacent classes

$$ZR_1\mathfrak{A},\ ZR_2\mathfrak{A},\ \dots,\ ZR_n\mathfrak{A},\ \dots$$

are conjugate in it, among them there can be only a finite number of distinct ones. Consequently, at least for one pair of elements ZR_i and ZR_j the relation

$$(ZR_i)(ZR_j)^{-1} \subset \mathfrak{A}$$

will hold. From this relation it follows that

$$ZR_iR_j^{-1}Z^{-1} \subset \mathfrak{A}.$$

Since the elements R_i and R_j are distinct from one another and are contained in the periodic group $\mathfrak{R}$, the element $R_iR_j^{-1}$ has finite order distinct from the identity. But then the element $ZR_iR_j^{-1}Z^{-1}$ of the subgroup $\mathfrak{A}$, which is conjugate to it, will also have finite order distinct from the identity. This, however, is impossible, since $\mathfrak{A}$ is a torsion-free group. Consequently, the group $\mathfrak{G}$ has no infinite classes of conjugate elements. The theorem is proved.

3. From the proof of the theorem one also immediately sees the validity of the following proposition.

Theorem 2. *If a nonperiodic group $\mathfrak{G}$ is an extension of its center by a locally finite group, then its elements of finite order form a subgroup (obviously invariant). The factor group of the group $\mathfrak{G}$ by this subgroup is an abelian torsion-free group.*

From Theorems 1 and 2 it follows:

Corollary 1 (4). *If all classes of conjugate elements of a nonperiodic group $\mathfrak{G}$ are finite, then its elements of finite order form a periodic subgroup. The factor group of the group $\mathfrak{G}$ by this subgroup is an abelian torsion-free group, and, in particular, if $\mathfrak{G}$ is torsion-free, then it is itself abelian.*

Corollary 2. A group $\mathfrak{G}$ has no infinite classes of conjugate elements if and only if it is either locally normal, or is an extension of a torsion-free group by means of some locally normal group having a periodic commutator subgroup.

Proof. The necessity of the conditions of this proposition follows directly from Theorems 1 and 2. Sufficiency is established as follows. If the group $\mathfrak{G}$ is locally normal, then all classes of its conjugate elements are, obviously, finite. If the group $\mathfrak{G}$ is an extension of a torsion-free subgroup $\mathfrak{A}$ by means of a locally normal group and its commutator subgroup is periodic, then the commutator $[X, Y] = X^{-1}Y^{-1}XY$, where X is an arbitrary element of $\mathfrak{A}$, and Y is an arbitrary element of $\mathfrak{G}$, will be an element of finite order contained in the subgroup $\mathfrak{A}$. But then, in view of the absence in the subgroup $\mathfrak{A}$ of elements of finite order distinct from the identity, $[X, Y] = 1$. Consequently, the subgroup $\mathfrak{A}$ is contained in the center of the group $\mathfrak{G}$. Since the factor group $\mathfrak{G}/\mathfrak{A}$ is locally normal, it follows, using Theorem 1, that the group $\mathfrak{G}$ has no infinite classes of conjugate elements.

Molotov State University
named after A. M. Gorky

Received
3 X 1957

CITED LITERATURE

[1] O. Yu. Shmidt, *Matem. sborn.*, **31**, 366 (1924).
[2] A. G. Kurosh, *Theory of Groups*, Moscow, 1953.
[3] Yu. G. Fedorov, *Uspekhi Mat. Nauk*, **6**, 1, 187 (1951).
[4] B. H. Neumann, *Proc. London Math. Soc.*, (3), **1**, 178 (1951).

Note: Figure translations are in progress. See original paper for figures.