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Abstract

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MATHEMATICS

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SOLUTION OF A MIXED PROBLEM BY THE FOURIER METHOD FOR AN INTEGRO-DIFFERENTIAL EQUATION

(Presented by Academician S. L. Sobolev on 26 X 1956)

In the present note an equation of the form

$$\frac{\partial^2 u}{\partial t^2} = Lu + a(X,t)u + d(X,t)\frac{\partial u}{\partial t} + b(X,t) + \int_\Omega K_0(X,Y,t)u(Y,t)\,d\Omega + \int_\Omega K_1(X,Y,t)\frac{\partial u}{\partial t}\,d\Omega, \tag{1}$$

is considered, where

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{i,j}(X)\frac{\partial u}{\partial X_j}\right] - c(X)u$$

is an operator whose coefficients are measurable and bounded in the finite connected domain Ω of variation of the point $X(x_1, x_2, \dots x_n)$ and satisfy in $\overline{\Omega}$ the conditions

$$c(X) \geq 0; \qquad a_{i,j} = a_{j,i}; \qquad \sum_{i,j=1}^{n} a_{i,j}\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2, \qquad \alpha = \text{const} > 0, \tag{2}$$

and the free term $b(X,t)$ and the coefficients $a(X,t), d(X,t)$ are defined in the cylinder $Q_l = \Omega \times (0 \leq t \leq l)$, $l < \infty$*. The functions $K_i(X,Y,t)$, $i = 0, 1$, are assumed to be defined in the domain $R_l = \Omega \times \Omega \times (0 \leq t \leq l)$.

For equation (1) the following mixed problem is posed:
Determine a solution of equation (1) satisfying the initial conditions

$$u\Big|_{t=0} = \varphi(X), \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(X) \tag{3}$$

and the boundary condition

$$u\big|_S = 0 \qquad \text{for } t \in [0, l], \tag{4}$$

where S is the boundary of the domain Ω.

Following O. A. Ladyzhenskaya, we shall call a generalized solution of the problem a function $u(X,t)$ which belongs to $D_1^0(Q_l)$ and satisfies the integral identity

$$\int_0^l \int_\Omega \left[\frac{\partial u}{\partial t}\frac{\partial \Phi}{\partial t} - \sum_{i,j=1}^{n} a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial \Phi}{\partial x_j} \right.$$

$$\left. -c(X)u\Phi + a(X,t)u\Phi + d(X,t)\frac{\partial u}{\partial t}\Phi + b(X,t)\Phi + \Phi\int_\Omega K_0(X,Y,t)u(Y,t)\,dY + \Phi\int_\Omega K_1(X,Y,t)\frac{\partial u}{\partial t}\,dY \right] dQ+ \tag{5}$$

for any function $\Phi(X,t)$ from $D_2^0(Q_l)$ and assumes the initial value $\varphi(X)$ in the following sense:

$$\int_\Omega |u(X,\Delta t) - \varphi(X)|^2\,d\Omega \to 0 \quad \text{as } \Delta t \to 0. \tag{6}$$

The solution is sought in the form of a Fourier series

$$u(X,t) = \sum_{n=1}^{\infty} f_n(t)v_n(X), \tag{7}$$

where $v_n(X)$ are the generalized eigenfunctions of the operator L.

Using the Fourier method, one can prove the existence and uniqueness of the solution of the posed problem and estimate the order of its growth. In addition, one can justify the applicability of the Galerkin method for an approximate solution of the problem.

Theorem 1. If the coefficients $a_{i,j}$ and c are measurable and bounded in Ω, while $d(X,t)$ and $a(X,t)$ are in Q_l; $\varphi \in D^0(\Omega)$; $\psi \in L_2(\Omega)$; the functions

$$\iint_{\Omega\Omega} |K_i(X,Y,t)|^2\,dY\,dX, \quad i = 0, 1,$$

are bounded on $[0, l]$, and $b(X,t)$ belongs to $L_2(Q_l)$, then there exists a generalized solution of equation (1) under conditions (3) and (4).

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* We shall use the notation and concepts from the book (1).

The solution can be represented in the form of the series (7), which converges in $W_2^1(\Omega, t)$ uniformly with respect to $t \in [0, l]$.

The main idea of the proof is as follows: an auxiliary countable differential system of equations is constructed, and the solution of this system is taken as the functions $f_n(t)$. Then it is shown that the series defined in this way,

$$\sum_{n=1}^{\infty} f_n(t) v_n(X),$$

converges in $W_2^1(\Omega, t)$ and that its sum is a generalized solution of the problem.

The auxiliary system, which plays the main role in the proof, has the form

$$f_m''(t) + \lambda_m^2 f_m(t) = \sum_{k=1}^{\infty} \left[a_{m,k}(t) + g_{m,k}^0(t)\right] f_k(t) +$$

$$+ \sum_{k=1}^{\infty} \left[d_{m,k}(t) + g_{m,k}^1(t)\right] f_k'(t) + b_m(t); \tag{8}$$

$$f_m(0) = \int_\Omega \varphi(X) v_m(X)\, d\Omega, \qquad f_m'(0) = \int_\Omega \psi(X) v_m(X)\, d\Omega, \qquad m = 1, 2, \dots,$$

where

$$a_{m,k}(t) = \int_\Omega a(X,t) v_m(X) v_k(X)\, d\Omega;$$

$$d_{m,k}(t) = \int_\Omega d(X,t) v_m(X) v_k(X)\, d\Omega;$$

$$g_{m,k}^i(t) = \int_\Omega v_m(X) \int_\Omega K_i(X,Y,t) v_k(Y)\, dY\, dX,$$

$$b_m(t) = \int_\Omega b(X,t) v_m(X)\, d\Omega;$$

λ_n^2 is the eigenvalue corresponding to the generalized eigenfunction v_m.

The study of system (8) is carried out in the Banach space G_2 of sequences of functions $v(t)$, $(v_1(t), v_2(t), \dots)$, measurable on $[0, l]$, such that the function

$$\bar{v}(t) = \left(\sum_{k=1}^{\infty} |v_k(t)|^2 \right)^{1/2}$$

is summable on $[0, l]$.

It is shown that system (8) has a unique solution in G_2; moreover, one can estimate the order of its growth, which makes it possible to estimate the norm of the solution of equation (1) in the space $W_2'(\Omega, t)$. As a result one obtains a rather cumbersome expression, which we do not present here. We note only that the correctness of the posed problem follows from this estimate.

For an approximate solution of the problem one may use the Galerkin method. The system of equations for determining the m-th aggregate u_m, obtained by the Galerkin method, has the form

$$\int_\Omega \left[\frac{\partial^2 u_m}{\partial t^2} v_k + \sum_{i,j=1}^{n} a_{i,j} \frac{\partial u_m}{\partial x_i} \frac{\partial v_k}{\partial x_j} + c(X) u_m v_k + a(X,t) u_m(X,t) v_k + \right.$$

$$+ d(X,t) \frac{\partial u_m}{\partial t} v_k + v_k \int_\Omega K_0(X,Y,t) u_m(Y,t)\, dY +$$

$$\left. + v_k \int_\Omega K_1(X,Y,t) \frac{\partial u_m}{\partial t}\, dY \right] d\Omega = 0, \qquad k = 1, 2, \dots, m. \tag{9}$$

If, as the complete orthonormal system in the Galerkin method, the system $\{v_m\}$ of eigenfunctions of the operator L is taken, then system (9) turns out to be truncated with respect to system (8). This makes it possible to prove the following theorem.

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Then, if* $\{u_m\}$ *is the sequence of aggregates obtained by the Galerkin method (with respect to the system* $v_k(X)$*), then*

$$\|u(X,t) - u_m(X,t)\|_D \to 0, \qquad \left\| \frac{\partial u}{\partial t} - \frac{\partial u_m}{\partial t} \right\|_{L_2(\Omega)} \to 0$$

uniformly with respect to $t \in [0, l]$.

Theorem 3 *(uniqueness theorem). If the coefficients* $a_{i,j}, c$ *are measurable and bounded in* Q_l*;* $d(X,t)$*,* $a(X,t)$ *are bounded in* Q_l*, and the functions*

$$\iint_{\Omega\Omega} |K_i(X,Y,t)|^2\, dY\, dX, \qquad i = 0, 1,$$

are bounded on $[0, l]$*, then the generalized solution of the homogeneous equation*

$$\frac{\partial^2 u}{\partial t^2} = Lu + au + d\frac{\partial u}{\partial t} + \int_\Omega K_0(X,Y,t)$$

$$u(Y,t)\,dY + \int_\Omega K_1(X,Y,t)\frac{\partial u}{\partial t}\,dY$$

with zero boundary and initial conditions is equivalent to zero.

The proof of Theorem 3 is based on the following lemma.

Lemma. *Let the conditions of Theorem 1 be satisfied and let the function $u(X,t)$ be a generalized solution of the problem. Then its Fourier coefficients with respect to the system v_m are absolutely continuous together with the first derivatives and satisfy almost everywhere on $[0,l]$ the infinite system of equations (8).*

It is interesting to note that uniqueness of the solution holds under the same conditions as existence.

In conclusion, let us note one particular case of the problem considered. For $K_i(X,Y,t) \equiv 0,\ i = 0,1$, we obtain the equation

$$\frac{d^2 u}{dt^2} = Lu + a(X,t)u + d(X,t)\frac{du}{dt} + b(X,t). \tag{10}$$

The justification of the Fourier method for solving the mixed problem for this equation, as far as we know, is carried out here for the first time. In the case $a = d = 0$, i.e., when the equation admits separation of variables, the Fourier method was justified by O. A. Ladyzhenskaya ([2,3]). A solution of the mixed problem for a general hyperbolic equation by the method of finite differences is given in the book ([1]). From the results obtained there it follows (as applied to the particular case in which the equation has the form (10)) that a generalized solution exists if the functions a_{ij}, $\partial a_{i,j}/\partial x_i$, c, a, and d are continuous in Q_1, $\varphi \in D^0(\Omega)$, $\psi \in L_2(\Omega)$, $b(X,t) \in L_2(Q_1)$.

The solution is unique if: the coefficients $a_{i,j}$ are continuous and have bounded generalized first-order derivatives of the form $\partial a_{i,j}/\partial x_i$, $\partial a_{i,j}/\partial t$, and second-order derivatives of the form $\partial^2 a_{i,j}/\partial x_i \partial x_j$; $d(X,t)$ is bounded in Q_1 and has the bounded generalized derivative $\partial d/\partial t$; $a(X,t)$ and $c(X)$ are measurable and bounded (respectively in the domains Q_1 and Ω); $b(X,t) \in L_2(Q_1)$; $\psi \in L_2(\Omega)$; $\varphi \in D^0(\Omega)$.

Comparison of these results with Theorems 1 and 3 (for $K_i \equiv 0$) shows that the Fourier method makes it possible to prove the existence and uniqueness of a solution of equation (10) under somewhat more general assumptions than the finite-difference method.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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