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Abstract

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MATHEMATICS

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ON A CERTAIN REPRESENTATION OF SOLUTIONS OF THE EULER-POISSON-DARBOUX EQUATION

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From the work ([1]) it follows that there is the possibility of representing, in the form

$$\gamma\left(\frac{c}{2}\right)\int_0^1 \frac{\varphi[x+iy(1-2\sigma)]\,d\sigma}{[\sigma(1-\sigma)]^{1-c/2}}, \qquad \gamma\left(\frac{c}{2}\right)=\frac{\Gamma(c)}{\Gamma^2(c/2)} \tag{1}$$

(where $\varphi(z)$ is an analytic function), any analytic solution of the equation

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{c}{y}\frac{\partial w}{\partial y}=0, \quad c=\text{const},\ c>0, \tag{2}$$

in the case when the solution is given in a convex domain symmetric with respect to the axis Ox.

In the present article we consider the possibility of representing, in the form (1), an arbitrary solution of equation (2) given only in a domain adjacent to the axis Ox. In this case the representation is possible if the solution is continuous on the axis Ox for $c \geq 1$, and for $0<c<1$, in addition, satisfies the condition

$$\lim_{y\to 0} y^c \frac{\partial w}{\partial y}=0. \tag{3}$$

Let us denote by T a simply connected domain lying in the half-plane $y>0$ and adjacent to an interval L of the axis Ox, and by $\overline{T}$ the domain symmetric to it with respect to the axis Ox.

We shall say that the domain T (or $\overline{T}$) belongs to the class B if the domain $T\cup L\cup\overline{T}$ contains entirely the segment joining any two of its points having common abscissas.

Consider the class $C_2(T)$ of functions continuous in $T\cup L$, possessing continuous first and second derivatives in T. Consider also the class of functions $N_2(T)$, to which we assign those functions of the class $C_2(T)$ that satisfy condition (3) on L. The classes $C_2(\overline{T})$ and $N_2(\overline{T})$ of functions given in $\overline{T}$ are defined correspondingly.

Theorem. *Every solution $w(x,y)$ of the class $C_2(T)$ of equation (2) for $c \geq 1$, and every solution of the class $N_2(T)$ of equation (2) for $0 < c < 1$, is represented in $T \in B$ in the form (1), where $\varphi(z)$ is a function of one complex variable analytic in $T \cup L \cup \overline{T}$, possessing on L the property $w(x,0) = \varphi(x)$.*

Proof. In the domain T consider an open semicircle τ of radius ρ with center at the point $(x_0, 0)$, adjacent to the interval ab (or l) of the axis Ox and bounded, for $y > 0$, by the semicircle γ. Using the values of the solution $w(x,y)$ on γ, construct in τ a bounded solution of equation (2), continuous in $\tau \cup \gamma$. In doing so, without loss of generality, we assume $w(a,0) = w(b,0) = 0$.

$w(x,y)$ on γ is a function $f(t)$, $t = \dfrac{x - x_0}{\rho}$, which is defined on the interval $[-1,1]$, is equal to zero at the endpoints, and on each interior interval

$$[-1+\delta,\, 1-\delta], \quad 0 < \delta < 1/2, \tag{4}$$

satisfies a Lipschitz condition.

Consider the function $f_\varepsilon(t)$, which coincides with $f(t)$ on the interval $[-1 + 3\varepsilon,\, 1 - 3\varepsilon]$, is equal to zero for $t \in \{[-1, -1+2\varepsilon], [1-2\varepsilon, 1]\}$, and at the remaining points takes such values that on the whole interval $[-1,1]$ it forms a k-times ($k = [\beta] + 1$) differentiable function.

The system of Gegenbauer polynomials $C_n^\beta(t)$ $(n = 0, 1, 2, \ldots)$ [(2)] is defined on the interval $[-1,1]$, is complete, and is orthogonal on it with weight $(1-t^2)^{\beta - 1/2}$. From Szegő' s work [(3)] it follows that, for $\beta > 0$, $f_\varepsilon(t)$ can be expanded in the series

$$\sum_{n=0}^{\infty} a_n C_n^\beta(t) \left(a_n = \frac{\Gamma^2(\beta)(n+\beta)\Gamma(n+\beta)}{2^{1-2\beta}\pi\Gamma(n+2\beta)} \int_{-1}^{+1} f_\varepsilon(x) C_n^\beta(x)(1-x^2)^{\beta-1/2}\,dx \right), \tag{5}$$

which converges uniformly on each interval (4).

The system

$$w_0^\beta = 1, \quad w_n^\beta(x,y) = \gamma(\beta) \int_0^1 \frac{[x + iy(1-2\sigma) - x_0]^n\,d\sigma}{[\sigma(1-\sigma)]^{1-\beta}} \quad (n = 1, 2, \ldots) \tag{6}$$

defines, in the entire plane of the variables x and y, a set of real functions, symmetric in y, that are solutions of equation (2) for $c = 2\beta$. On the semicircle

γ the functions of the system (6) coincide, up to constant factors, with the system of Gegenbauer polynomials $C_n^\beta(t)$ $\left(t = \dfrac{x - x_0}{\rho}\right)$:

$$w_n^\beta(x, y)\Big|_{\substack{x=x_0+\rho t \\ y=\rho\sqrt{1-t^2}}} = \rho^n \psi_n C_n^\beta(t), \quad \text{where} \;\; \psi_n = \frac{\Gamma(2\beta)\Gamma(n+1)}{\Gamma(2\beta + n)}.$$

Using the coefficients of the expansion (5), we form the expression

$$\sum_{n=0}^{\infty} b_n \gamma(\beta) \int_0^1 \frac{[x + iy(1 - 2\sigma) - x_0]^n \, d\sigma}{[\sigma(1-\sigma)]^{1-\beta}}, \quad b_n = \frac{a_n}{\rho^n \psi_n}. \tag{7}$$

Extend it to complex values $x = \dfrac{z+\zeta}{2}$, $y = \dfrac{z-\zeta}{2i}$ in the form

$$\sum_{n=0}^{\infty} b_n \gamma(\beta) \int_0^1 \frac{[(z - x_0)(1-\sigma) + (\zeta - x_0)\sigma]^n \, d\sigma}{[\sigma(1-\sigma)]^{1-\beta}}. \tag{8}$$

Using the asymptotic expansion of the polynomials $C_n^\beta(t)$ for large n [(4)] and taking into account that, as $n \to \infty$, $\Gamma(n + a) = O(n^a)\Gamma(n)$, we obtain the estimates $a_n = O(n^\beta)$, $\psi_n = \Gamma(2\beta)n^{1-2\beta} + o(n^{1-2\beta})$. Therefore the series (8) converges uniformly in any bicylinder $|z - x_0| \le q\rho$, $|\zeta - x_0| \le q\rho$ $(0 < q < 1)$.

Since each integral in expression (8) is a polynomial in powers of z and ζ, by Weierstrass' theorem for analytic functions of several complex variables ([(5)], pp. 29 and 325), we obtain that the sum of the series (8) is a single-valued analytic function in the bicylindrical domain $|z - x_0| \le \rho$, $|\zeta - x_0| \le \rho$. The series (7), which is the series (8) for the values $\zeta = \bar{z}$ and $z = x + iy$, converges uniformly in any

closed disk $\sqrt{(x - x_0)^2 + y^2} \le q\rho$ $(0 < q < 1)$ and determines in $\tau \cup l \cup \bar{\tau}$ a certain real, analytic, and y-symmetric solution $w_\varepsilon(x, y)$ of equation (2).

The series (7) can be represented in the form

$$\sum_{n=0}^{\infty} a_n \lambda^n C_n^\beta(t), \qquad \lambda = \frac{\sqrt{(x - x_0)^2 + y^2}}{\rho}, \qquad t = \frac{x - x_0}{\rho}. \tag{9}$$

Since the series (5) converges uniformly on any segment (4), and the sequence λ^n is nonincreasing, by Abel' s test for uniform convergence of functional series (6) we obtain that the series (9) converges uniformly for $t \in [-1 + \delta, 1 - \delta]$ and $\lambda \le 1$. Thus the sum of the series (7) is continuous at every point of the semicircle γ and assumes on it the values of the function $f(t)$.

The partial sum of the series (5) has the expression (7)

$$S_n(t) = O(1) \int_{-1}^{+1} f_\varepsilon(x) \frac{p_{n+1}(x)p_n(t) - p_n(x)p_{n+1}(t)}{x-t} (1-x^2)^{\beta - \frac{1}{2}} \, dx, \qquad (10)$$

where $p_n(t)$ are the orthonormal Gegenbauer polynomials

$$p_n(t) = \frac{\sqrt{n+\beta}}{2^{\beta+\frac{1}{2}}} \frac{\Gamma(2\beta)}{\Gamma(\beta+\frac{1}{2})} \frac{\sqrt{\Gamma(n+1)}}{\sqrt{\Gamma(n+2\beta)}} C_n^\beta(t) \qquad (n = 0, 1, 2, ...).$$

In view of the fact that (8)

$$|C_n^\beta(t)| \leq |C_n^\beta(\pm 1)| = \frac{1}{\Gamma(2\beta)} \frac{\Gamma(n+2\beta)}{\Gamma(n+1)},$$

as $n \to \infty$ we have $p_n(t) = O(n^\beta)$, $t \in [-1, 1]$.

The asymptotic expansion of the Jacobi polynomials obtained by V. A. Steklov ([4]), for $0 < \theta < \pi$, $t = \cos\theta$, gives for $p_n(t)$ the expression

$$p_n(t) = O(1) \sin^{-\beta}\theta \left\{ \cos\left[(n+\beta)\theta + \frac{\beta}{2}\pi \right] + \frac{\vartheta_n}{n+\beta} \right\}, \qquad \text{where } \vartheta_n < A \sin^{-1}\theta,$$

with the aid of which we consider (10) for the values

$$t \in \{[-1, -1+\varepsilon], \ [1-\varepsilon, 1]\}. \qquad (11)$$

Integrating $S_n(t)$ k times by parts on the interval $[-1+2\varepsilon, 1-2\varepsilon]$, we obtain $S_n(t) = O(n^{\beta-k-1})$, i.e., $S_n(t)$ tends uniformly to zero for all t belonging to (11).

By the Abel test cited above, the series (9) converges uniformly for t belonging to (11) and $\lambda \leq 1$. The latter means that $w_\varepsilon(x, y)$ is continuous at the points a and b. From works ([9], [10]) there follows the following maximum principle: for every bounded solution $w(x, y)$ of equation (2), belonging to the class $C_2(\tau)$ for $c \geq 1$ or to the class $\widetilde{N}_2(\tau)$ for $0 < c < 1$, continuous in the closure of τ (i.e., $\tilde{\tau}$), the inequality

$$\inf_\gamma w(x, y) \leq w(x, y) \leq \sup_\gamma w(x, y)$$

holds for all points τ.

Moreover, every such solution in $\tilde{\tau}$ can be represented by means of the Green function ([9], [11]); therefore any infinite bounded family of solutions $\{w(x, y)\}$ is compact in $\tilde{\tau}$.

Choose a sequence $\varepsilon_n \to 0$ as $n \to \infty$. Construct a sequence of functions $f_{\varepsilon_n}(t)$ on γ and a sequence of solutions $w_{\varepsilon_n}(x, y)$ in $\tilde{\tau}$.

As $n \to \infty$, $w_{\varepsilon_n}(x, y)$ converge uniformly in $\tilde{\tau}$ to a limiting function $w^*(x, y)$, which will be a bounded solution of equation (2), belonging to the class $N_2(\tau)$, continuous in $\tilde{\tau}$, and taking on γ the values $f(t)$. By the maximum principle we have $w^*(x, y) \equiv w(x, y)$ everywhere in $\tilde{\tau}$.

Since $w^*(x, y)$ is analytic in $\tau \cup l \cup \bar{\tau}$, it follows, by (1), that it is represented there in the form (1). Moreover, the representation is unique. Hence $w(x, y)$ is represented in τ in the form (1) in a unique way.

Consider the domain σ adjacent to L, composed of the totality of semicircles τ_k $(k = 1, 2, ...)$. On the basis of the uniqueness of the representation of $w(x, y)$ by means of the analytic function $\varphi(z)$ in each τ_k, we obtain that $w(x, y)$ is represented in σ in the form (1).

Taking into account that $w(x, y)$ is analytically continued into the domain $\bar{\sigma}$, and therefore also into $\bar{T}$, on the basis of work [1] we obtain the representability of $w(x, y)$ in any domain $T^* \in T$ such that the domain $T^* \cup L \cup \overline{T^*}$ is convex.

Since any domain $T \in B$ can be represented as the sum of a countable number of domains T_n^*, we make certain that the solution $w(x, y)$ is represented in the form (1) throughout the whole domain T.

Corollary. Every solution of class $C_2(T)$ $(T \in B)$ of equation (2) for $c \geqslant 1$, and every solution of class $N_2(T)$ $(T \in B)$ of equation (2) for $0 < c < 1$:

a) is analytic in x and y in $T \cup L$ and can be extended symmetrically with respect to y into the conjugate domain $\bar{T}$, forming in $T \cup L \cup \bar{T} = D$ an analytic solution of equation (2);

b) is analytically continued to complex values z and ζ in the form $w\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$ into the bicylindrical domain $z \times \zeta = D \times D$.

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Note: Figure translations are in progress. See original paper for figures.