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Abstract

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MATHEMATICS

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INVESTIGATION OF BOUNDARY-VALUE PROBLEMS FOR ELLIPTIC SYSTEMS OF DIFFERENTIAL EQUATIONS IN THE PLANE

(Presented by Academician I. G. Petrovskii, 11 XII 1956)

The paper is devoted to the investigation of boundary-value problems for elliptic systems of equations of first order, to which, in an equivalent manner (in the sense indicated below), the following boundary-value problem is reduced.

Problem 1. Find a solution $U(z)$ of class K_n of the elliptic system

$$\sum_{k+l\leq n}{}' A_{kl}(z)\frac{\partial^{k+l}U}{\partial x^k \partial y^l} = F(z) \qquad (z = x + iy \in D),$$

satisfying the boundary condition

$$\sum_{k+l\leq n-1}\left[a_{kl}(t)\,U^+_{kl}(t) + \int_\Gamma b_{kl}(t,t_1)\,U^+_{kl}(t_1)\,ds_1\right] = f(t) \qquad (t \in \Gamma).$$

Here $A_{kl}(z)$ are real square matrices of order p, defined in a certain domain $\bar{D}$, which are assumed to be Hölder continuous for $k + l < n$ and to have first Hölder-continuous derivatives for $k + l = n$; $a_{kl}(t)$ and $b_{kl}(t, t_1)$ are matrices of size $\left(\frac{np}{2} \times p\right)$, where

$$b_{kl}(t,t_1) = \frac{\tilde{b}_{kl}(t,t_1)}{|t-t_1|^\alpha} \qquad (0 \leq \alpha < 1);$$

$\tilde{b}_{kl}(t, t_1)$, $a_{kl}(t)$ $(0 \leq k + l \leq n - 1)$ satisfy a Hölder condition on Γ; $U^+_{kl}(t)$ is the limiting value of the derivative $\partial^{k+l}U/\partial x^k \partial y^l$ from inside the domain D; ds_1 is the element of arc at the point $t_1 \in \Gamma$; D is a finite simply connected domain with boundary Γ, smooth in the Hölder sense; $D + \Gamma \subset \bar{D}$; $n \geq 1$ is the order of the highest derivatives of all the unknown functions entering the system; K_n is the class of functional columns having n-th continuous derivatives in D and

$(n-1)$-st continuous derivatives in $D+\Gamma$, satisfying a Hölder condition on Γ. Ellipticity is understood in the sense of I. G. Petrovskii ([7,8]).

By the substitution

$$\frac{\partial^{n-1}U}{\partial x^{k-1}\partial y^{n-k}} = u_k \quad (k=1,2,\dots,n), \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \tag{1}$$

with the use of the integral identity of S. L. Sobolev $((^9), pp.57$—$62)$, under the assumption that the domain D is star-shaped with respect to some circle, problem 1 is reduced to the following boundary-value problem:

Problem 2. Find a column u of height $2r = np$ and a constant column c of height $r(n-1)$, which satisfy the system

$$\mathscr{L}u \equiv A(z)\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + B(z)u + \iint_D R(z,\zeta)u(\zeta)\,d\xi d\eta = \widetilde{F}(z) + M(z)c \tag{2}$$

$$(z = x+iy \in D, \quad \zeta = \xi + i\eta)$$

and the boundary condition

$$\Lambda u \equiv a(t)u(t) + \int_\Gamma b(t,t_1)u(t_1)\,ds_1 = \tilde{f}(t) + N(t)c \qquad (t \in \Gamma). \tag{3}$$

Here and below, by solutions of systems of the form $\mathscr{L}u = F$ we mean solutions of class K_1; the right-hand sides of the system of equations and of the boundary conditions are assumed to be continuous in the Hölder sense.

Problems 1 and 2 are equivalent in the sense that a one-to-one correspondence between the solutions of these problems is established by the equalities (1) and the integral identity of S. L. Sobolev (c is the column composed of the coefficients of the polynomials entering this identity).

By a linear smooth nonsingular transformation, system (2) is reduced to a form in which

$$A(z) = \begin{pmatrix} A^{(1)}(z) & -A^{(2)}(z) \\ A^{(2)}(z) & A^{(1)}(z) \end{pmatrix},$$

and for all eigenvalues $\lambda(z)$ of the matrix $A^{(1)}(z) + iA^{(2)}(z)$ the condition $\operatorname{Im}\lambda(z) > 0$ $(z \in \widetilde{D})$ is satisfied. It is proved that the ellipticity condition is sufficient for the possibility of such a reduction.

In studying problem 2, the methods of I. N. Vekua (^{1,2}) are used. The investigation is carried out with the aid of fundamental matrices constructed by Ya. B. Lopatinskii (^{3,4}).

Taking into account that

$$u(z) = \iint_D \varphi(\zeta, z)[\widetilde{F}(\zeta) + M(\zeta)c]\, d\xi d\eta,$$

where $\varphi(\zeta, z)$ is the fundamental matrix of the operator $\mathcal{L}$; $z = x + iy \in D$, $\zeta = \xi + i\eta$, being a solution of system (2), we reduce problem 2 to the form

$$\mathcal{L}u = 0, \qquad \Lambda u = f_0 \tag{4}$$

(the column c is included in f_0).

Theorem 1. Each solution $u(z)$ of the system $\mathcal{L}u = 0$ is representable in the form

$$u(z) = \int_\Gamma M(\zeta, z)\mu(\zeta)\, d\zeta_s + \sum_{j=1}^{r} c_j u^{(j)}(z), \tag{5}$$

where $\mu(\zeta)$ is a column of height r, satisfying the Hölder condition on Γ; c_j ($j = 1, \dots, r$) are constants. The column $\mu(\zeta)$ and the constants c_j are determined uniquely by $u(z)$.

Here

$$M(\zeta, z) = \frac{1}{2\pi} \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} [(\xi - x) + A(z)(\eta - y)]^{-1} \times$$

$$\times [A(\zeta)\cos(\nu\xi) - \cos(\nu\eta)] \begin{pmatrix} E \\ 0 \end{pmatrix} + \gamma(\zeta, z),$$

where $\zeta = \xi + i\eta$; $z = x + iy$; ν is the inward normal to Γ; E is the identity matrix of order r. The matrix $\gamma(\zeta, z)$ contains the lower (in order of singularity) terms of the fundamental matrix and some bounded ⋯

summands, which can be constructed by the method indicated by Ya. B. Lopatinskii (^{5}). $u^{(j)}(z)$ ($j = 1, \dots, r$) are particular solutions of the system under consideration; $2r$ is the number of equations in the system.

With the aid of representation (5), problem (4) is reduced to an equivalent system of singular integral equations with a Cauchy-type kernel. It is assumed that the condition

$$\det[a_1(t) + ia_2(t)] \neq 0 \qquad (t \in \Gamma),$$

where $(a_1(t), a_2(t)) = a(t)$ (see (3)), is satisfied. Then the resulting system of equations is of normal type, and its index $\varkappa$, computed by the formula of N. I. Muskhelishvili (^{6}), is equal to

$$\varkappa = \frac{1}{\pi}\,[\arg\det(a_1 + ia_2)]_\Gamma. \tag{6}$$

It follows from the preceding that the homogeneous problem

$$\mathcal{L}u = 0, \qquad \Lambda u = 0 \tag{7}$$

has a finite number of linearly independent solutions.

Theorem 2. For the solvability of problem (4) for an arbitrary right-hand side f_0, it is necessary and sufficient that the corresponding homogeneous problem (7) have $\varkappa + r$ linearly independent solutions.

The adjoint boundary-value problem is formulated. It is assumed here that the system of r equations $\Lambda u = 0$ can be completed to a system of $2r$ homogeneous Fredholm equations having no nonzero solutions (for $b(t, t_1) \equiv 0$, no additional restrictions are imposed on Λ).

Theorem 3. The difference between the number of linearly independent solutions of problem (7) and of the homogeneous adjoint problem is equal to $\varkappa + r$ (see (6)).

Theorem 4. For the solvability of the problem

$$\mathcal{L}u = F, \qquad \Lambda u = 0$$

it is necessary and sufficient that the right-hand side F be orthogonal to all solutions of the homogeneous adjoint problem.

When the condition of Theorem 2 is fulfilled, a normalization of the solutions is introduced so that problem (4), in the class of normalized solutions, turns out to be uniquely solvable for any right-hand side f_0. A continuous dependence of the solutions of problem (4) on the right-hand side is proved in the following sense. Let $f_n \to f_0$ in the metric H^α ((^{6}), § 49). Then the normalized solutions u_n of the problem $\mathcal{L}u_n = 0$, $\Lambda u_n = f_n$ converge uniformly in $D + \Gamma$ to the normalized solution u of problem (4), and the first derivatives of u_n converge uniformly to the corresponding first derivatives of u inside D (i.e., in every closed subdomain).

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Note: Figure translations are in progress. See original paper for figures.

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