# MATHEMATICS

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## Abstract

## Full Text

# MATHEMATICS

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# On Mean Moduli of Analytic Functions

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Denote by $S(\alpha, \beta)$ the strip $\alpha < \operatorname{Re} z < \beta$ of finite width, and by $S[\alpha, \beta]$ the strip $\alpha \leqslant \operatorname{Re} z \leqslant \beta$. Hardy, Ingham, and Pólya ($^1$) considered means of the form

$$\varphi_p(x, y) = \frac{1}{2y} \int_{-y}^{y} |f(x + i\eta)|^p \, d\eta \quad (p > 0), \tag{1}$$

where the functions $f(z) = f(x + iy)$ satisfy the following condition E:

a) $f(z)$ is analytic in $S(\alpha, \beta)$ and continuous in every finite part of $S[\alpha, \beta]$;

b) $f(z) = O(e^{k|y|})$ uniformly in $S(\alpha, \beta)$, $0 < k < \dfrac{\pi}{\beta - \alpha}$.

The present paper contains analogous theorems in which the functions are free of the restriction of continuity on the boundary of the domain.

In what follows, as a rule, we shall take $\alpha = 0$ and $\beta = \pi$, although the results are valid in the general case. We shall always assume that the functions under consideration are analytic inside the strip.

**Definition 1.** $f(z) \in A$ (respectively $H_p$) if the subharmonic function $\ln^+ |f(z)|$ (respectively $|f(z)|^p$) has a harmonic majorant in $S$.

**Definition 2.** $f(z) \in \mathfrak{M}_p$ if $\varphi_p(x, y) \leqslant M(f) = \text{const}$ in $S$.

**Lemma 1.** Let $u(x, y)$ be a nonnegative subharmonic function in $S(0, \pi)$; let $\lambda(t) \geq 0$ be a continuous nondecreasing function ($t > 0$), and

$$\int_{-\infty}^{\infty} u(x, y) \lambda(|y|) \, dy < M = \text{const}.$$

Then for every $\delta$ ($0 < \delta < \pi/2$) in the strip $S(\delta, \pi - \delta)$ the following hold:

1) $\lambda(|y| + \delta) u(x, y) \to 0$ uniformly as $|y| \to \infty$;

2)

$$u(x, y) \leqslant \frac{M}{\delta \lambda(|y| + \delta)}.$$

**Lemma 2.** If a nonnegative subharmonic function in $S(0, \pi)$, $u(x+iy)$, satisfies the condition

$$\int_{-\infty}^{\infty} u(x+iy)e^{-|y|}\,dy < M(u) = \text{const} \quad (0 < x < \pi), \tag{2}$$

then it has a harmonic majorant in $S(0, \pi)$.

Indeed, by means of the substitution

$$\frac{w-1}{w+1} = ie^{iz} \tag{3}$$

$S(0, \pi)$ is mapped onto the disk $|w| < 1$, and condition (2) is transformed into

$$\int_{\gamma} u(w)\,|dw| \leqslant \text{const},$$

where $\gamma$ is the image of the line $\operatorname{Re} z = \text{const}$.

With the aid of the images of the lines $x = \delta$, $x = \pi - \delta$ and of line segments perpendicular to the $x$-axis, one can form a convex contour $\Gamma$ approximating the circle $|w| = 1$. On the basis of Lemma 1 we have

$$\int_{\Gamma} u(w)|dw| \leqslant 4\left(M + \frac{\varepsilon}{\operatorname{tg}\delta}\right) \qquad (\varepsilon > 0).$$

On the basis of Gabriel' s theorem ($^2$)

$$\int_{|w|=\rho<1} u(w)|dw| \leqslant 8\left(M + \frac{\varepsilon}{\operatorname{tg}\delta}\right).$$

Since $\varepsilon$ is arbitrary, we conclude (3), that $u(w)$ has a harmonic majorant in $|w| < 1$, and $u(z)$ in $S(0, \pi)$.

**Corollary.** $\mathfrak{M}_p \subset H_p$.

Indeed, let

$$\Phi(x, t) = \int_0^t \{|f(x+i\sigma)|^p + |f(x-i\sigma)|^p\}\,d\sigma.$$

Then

$$\int_{-\infty}^{\infty} e^{-|y|}|f(x+iy)|^p\,dy = \int_0^{\infty} e^{-y}\Phi(x, y)\,dy \leqslant 2M,$$

and on the basis of Lemma 2 we obtain the assertion.

**Theorem 1.** If $f(z) \in \mathfrak{M}_p$, then:

a) $f(z)$ has finite angular boundary values almost everywhere on the lines $x = 0$ and $x = \pi$;

b) these boundary values are summable to the power $p$ on every finite interval of variation of $y$; moreover

$$\varphi_p(\xi, y) \leqslant M(f) \qquad \text{for } \xi = 0 \text{ and } \xi = \pi;$$

c) $f(z) = B(z)g(z)$, where $B(z)$ is the Blaschke product for the strip and the zeros of $f(z)$:

$$B(z) = \operatorname{ctg}^{\lambda}\left(\frac{z}{2} + \frac{\pi}{4}\right) \prod_{k=1}^{\infty} \frac{\sin\dfrac{z - z_k}{2}}{\sin\dfrac{z + z_k}{2}} \left(\frac{\cos \overline{z}_k}{\cos z_k}\right)^{1/2};$$

$\lambda$ is the multiplicity of the point $z = \pi/2$ as a zero of $f(z)$, and the convergence condition is

$$\sum_{(z_k)} e^{-|y_k|} \sin x_k < +\infty;$$

$g(z) \neq 0$ in $S(0, \pi)$; $g(z) \in \mathfrak{M}_p$, and $g(z)$ has almost everywhere on the lines $x = 0$ and $x = \pi$ angular boundary values whose moduli are equal to $|f(z)|$;

d) if $E$ is any bounded measurable set in $-\infty < y < \infty$, then

$$\int_E |f(x + iy)|^{pdy} \to \int_E |f(\xi + iy)|^{pdy} \qquad \text{as } x \to \xi \ (\xi = 0 \text{ and } \xi = \pi);$$

d) for every bounded measurable set $E$

$$\int_E |f(x + iy) - f(\xi + iy)|^p \, dy \to 0 \quad \text{as } x \to \xi \ (\xi = 0 \text{ and } \xi = \pi).$$

Let us consider propositions that may be called theorems of the Phragmén-Lindelöf type for the means $\varphi_p(x, y)$. The proofs are based on the following integral representations of the classes $A$ and $H_p$.

**Lemma 3.** Every positive harmonic function in $S(0, \pi)$ is representable in the form

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin x \ \operatorname{ch} \eta}{\Delta_1} \, d\psi_0(\eta) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin x \ \operatorname{ch} \eta}{\Delta_2} \, d\psi_\pi(\eta)$$
$$+ (Me^{y} + Ne^{-y}) \sin x,$$

where

$$\Delta_1 = \operatorname{ch}(y-\eta) - \cos x, \qquad \Delta_2 = \operatorname{ch}(y-\eta) + \cos x;$$

$\psi_0(\eta)$ and $\psi_\pi(\eta)$ are nondecreasing functions bounded for $-\infty < \eta < \infty$; $M$ and $N$ are nonnegative constants determined by the relations

$$M = \lim_{y\to\infty} e^{-y} u\Big(\frac{\pi}{2}, y\Big), \qquad N = \lim_{y\to-\infty} e^{y} u\Big(\frac{\pi}{2}, y\Big).$$

**Theorem 2.** In order that $f(z) \in A$ (respectively $H_p$), it is necessary and sufficient that $f(z)$ can be represented in the form

$$f(z) = e^{i\lambda} \exp i\{Me^{iz} - Ne^{-iz}\} B(z) D(z) g(z), \tag{4}$$

where $\lambda, M, N$ are real constants; $B(z)$ is a Blaschke product;

$$D(z) = \exp \frac{1}{2\pi i} \left\{ \int_{-\infty}^{\infty} \ln p_0(\eta)\, \frac{e^{iz} + e^{\eta}}{(1 - e^{iz}e^{\eta}) \operatorname{ch}\eta}\, d\eta + \int_{-\infty}^{\infty} \ln p_\pi(\eta)\, \frac{e^{iz} - e^{\eta}}{(1 + e^{iz}e^{\eta}) \operatorname{ch}\eta}\, d\eta \right\},$$

$$p_0(\eta) \geq 0; \qquad p_\pi(\eta) \geq 0; \qquad \int_{-\infty}^{\infty} e^{-|\eta|} \ln p_0(\eta)\, d\eta < +\infty;$$

$$\int_{-\infty}^{\infty} e^{-|\eta|} \ln p_\pi(\eta)\, d\eta < +\infty;$$

$$g(z) = \exp \frac{1}{2\pi i} \left\{ \int_{-\infty}^{\infty} \frac{e^{iz} + e^{\eta}}{1 - e^{iz}e^{\eta}}\, d\psi_0(\eta) + \int_{-\infty}^{\infty} \frac{e^{iz} - e^{\eta}}{1 - e^{iz}e^{\eta}}\, d\psi_\pi(\eta) \right\};$$

$\psi_0(\eta)$ and $\psi_\pi(\eta)$ have bounded variation in $(-\infty, \infty)$ and derivative almost everywhere equal to zero.

For the classes $H_p$, in addition,

$$M \geq 0; \qquad N \geq 0; \qquad \int_{-\infty}^{\infty} e^{-|\eta|} [p_0(\eta)]^p\, d\eta < +\infty; \qquad \int_{-\infty}^{\infty} e^{-|\eta|} [p_\pi(\eta)]^p\, d\eta < +\infty;$$

$\psi_0(\eta)$ and $\psi_\pi(\eta)$ are nondecreasing functions.

**Theorem A.** In order that $f(z) \in H_p$ have bounded means $\varphi_p(x, y)$ in $S(\alpha, \beta)$, it is necessary and sufficient that

$$\varphi_p(\alpha, y) \leq K, \qquad \varphi_p(\beta, y) \leq K \quad (K = \text{const}). \tag{5}$$

Moreover, from (5) it follows that $\varphi_p(x, y) \leq K$ for $\alpha \leq x \leq \beta$.

Let us show the sufficiency of the conditions. From (4) it follows that

$$|f(x+iy)| \leq \exp \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \ln p_0(\eta) \frac{\sin x}{\Delta_1}\, d\eta \right\} \exp \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \ln p_\pi(\eta) \frac{\sin x}{\Delta_2}\, d\eta \right\}.$$

Applying Hölder' s inequality and the inequality between the arithmetic mean and the geometric mean, we obtain

$$\varphi_p(x,y) \le \left\{ \frac{1}{2y} \int_{-y}^{y} dt \int_{-\infty}^{\infty} [p_0(\eta)]^p \, \frac{\sin x}{2(\pi - x)\Delta_1} \, d\eta \right\}^{(\pi - x)/\pi} \times \left\{ \frac{1}{2y} \int_{-y}^{y} dt \int_{-\infty}^{\infty} [p_\pi(\eta)]^p \, \frac{\sin x}{2x\Delta_2} \, d\eta \right\}^{x/\pi}.$$

If we introduce the functions

$$F_\xi(\eta) = \int_0^\eta \{[p_\xi(\sigma)]^p + [p_\xi(-\sigma)]^p\} \, d\sigma \qquad (\xi = 0 \text{ and } \xi = \pi)$$

and integrate by parts, then we find that

$$\varphi_p(x,y) \le K \quad \text{in } S(0,\pi).$$

**Theorem B.** In order that a function $f(z)$ of class $A$ also belong to the class $\mathfrak{M}_p$, it is necessary and sufficient that the following conditions be satisfied:

a) the least harmonic majorant of the function $\ln^+|f(z)|$ is the limit of a uniformly convergent, inside $S(\alpha,\beta)$, minorizing sequence of harmonic functions bounded above;

b) $\varphi_p(\alpha,y) \le K, \quad \varphi_p(\beta,y) \le K \ (K = \text{const})$.

Moreover, from a) and b) it follows that $\varphi_p(x,y) \le K$ for $\alpha \le x \le \beta$.

The proof follows from the theorem of P. Ya. Polubarinova-Kochina ($^3$) and from ($^4$).

Let us note that functions $f(z) \in A$ satisfying condition a) of Theorem B in $S(0,\pi)$ are characterized also by the fact that in their integral representation (4) one has $M \ge 0$, $N \ge 0$; $\psi_0(\eta)$ and $\psi_\pi(\eta)$ are nonincreasing functions.

**Theorem C.** If $f(z) \in \mathfrak{M}_p$ in $S(\alpha,\beta)$, then

$$\varphi_p(x,y) \le A^{\frac{\beta - x}{\beta - \alpha}} B^{\frac{x - \alpha}{\beta - \alpha}},$$

where $\varphi_p(\alpha,y) \le A$ and $\varphi_p(\beta,y) \le B$, i.e. the logarithm of the upper bound of the means $\varphi_p(x,y)$ is a convex function of $x$ for $\alpha \le x \le \beta$.

The proof follows directly from Theorem A.

**Theorem D.** Suppose $f(z)$ satisfies condition a) of Theorem B and, in addition:

$$\text{b')} \quad \varlimsup_{y\to\infty} y^{-a} \int_{-y}^{y} |f(\alpha + i\eta)|^p \, d\eta \le A, \qquad \varlimsup_{y\to\infty} y^{-b} \int_{-y}^{y} |f(\beta + i\eta)|^p \, d\eta \le B, \quad a \ge 0,$$

$$b \ge 0, \quad A > 0, \quad B > 0.$$

Then

$$\varlimsup_{y\to\infty} y^{-\mu} \int_{-y}^{y} |f(x + i\eta)|^p \, d\eta \le M \qquad (\alpha \le x \le \beta),$$

where

$$\mu = a\frac{\beta - x}{\beta - \alpha} + b\frac{x-\alpha}{\beta-\alpha}, \qquad M = A^{\frac{\beta-x}{\beta-\alpha}} B^{\frac{x-\alpha}{\beta-\alpha}}.$$

In particular, if $\varphi_p(\alpha, y) \leq M,\ \varphi_p(\beta, y) \leq M$, then $\varphi_p(x, y) \leq M$ for $\alpha \leq x \leq \beta$, and

$$\ln\left[\overline{\lim_{y\to\infty}}\, \varphi_p(x, y)\right]$$

is a convex function of $x$.

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*Note: Figure translations are in progress. See original paper for figures.*