MATHEMATICS

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. V. Malyshev

ON THE DISTRIBUTION OF INTEGRAL POINTS ON THE FOUR-DIMENSIONAL SPHERE

(Presented by Academician I. M. Vinogradov on 8 XII 1956)

The Jacobi theorem is known (see, for example, (1)) on the number $r(n)$ of representations of a given positive integer n as a sum of four integer squares, i.e., on the number of all integral points (x_1, x_2, x_3, x_4), where x_1, x_2, x_3, x_4 are integers, on the four-dimensional sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = n$; namely:

$$r(n) = 24\sigma(n),$$

where $\sigma(n)$ is the sum of all odd divisors of the number n. Using the usual methods of analytic number theory, in particular the Kloosterman method (2), one can also prove the asymptotic uniformity of the distribution of integral points (x_1, x_2, x_3, x_4) on the sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = n$, where n is an odd number. Apart from its independent interest, this result has applications in the theory of ternary quadratic forms (see, for example, (3,4)). In the present note we shall outline the main points of the proof of this theorem.

Theorem 1. *Let* Ω *be a four-dimensional convex conical region* with vertex at the origin $(0,0,0,0)$ and solid angle $\omega > 0$; let n be an odd positive integer. Denote by $r(\omega, n)$ the number of integral points of the sphere $x_1^2+x_2^2+x_3^2+x_4^2 = n$ lying in the region Ω. Then, as $n \to \infty$ and for fixed ω,*

$$r(\omega, n) = \frac{\omega}{2\pi^2}\, r(n) \left(1 + O\left(n^{-1/18+\varepsilon}\right)\right).$$

Let us note that we allow Ω to depend on n, fixing only $\omega > 0$. One can obtain the asymptotic formula

$$r(\Omega, n) \sim \frac{\omega}{2\pi^2}\, r(n)$$

for an arbitrary cone Ω, quadrable in the sense of Jordan, if it is regarded as fixed, independent of n. We also note that Theorem 1 includes, in particular, Wright' s result (5).

We give a sketch of the proof of Theorem 1.

1°. Let l_1, l_2, l_3 be integers satisfying the condition $\sqrt{l_1^2+l_2^2+l_3^2} \leq n^{1/16}$. Put

$$r_{l_1 l_2 l_3}(n) = \sum_{x_1^2+x_2^2+x_3^2+x_4^2=n} \exp\left[2\pi i\left(\frac{l_1}{\sqrt{n}}x_1 + \frac{l_2}{\sqrt{n}}x_2 + \frac{l_3}{\sqrt{n}}x_3\right)\right].$$

* A conical region Ω is a set of points $(\xi_1, \xi_2, \xi_3, \xi_4)$ with the property that if $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega$ and $\lambda > 0$, then $(\lambda\xi_1, \lambda\xi_2, \lambda\xi_3, \lambda\xi_4) \in \Omega$; in particular, the whole space is a conical region with solid angle $2\pi^2$.

Then, generalizing and refining Kloosterman' s arguments $^{(2)}$, we can obtain, as $n \to \infty$, the following asymptotic formula:

$$r_{l_1 l_2 l_3}(n) = \pi^2 n S(n) \frac{J_1\left(2\pi\sqrt{l_1^2+l_2^2+l_3^2}\right)}{\pi\sqrt{l_1^2+l_2^2+l_3^2}} + O\left(n^{17/18+\varepsilon}\right), \tag{1}$$

where the constants entering into O do not depend on l_1, l_2, l_3. Here

$$J_2(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+1}}{k!(k+1)!}$$

is the Bessel function; in order not to exclude the case

$$l_1 = l_2 = l_3 = 0,$$

when $r_{l_1 l_2 l_3}(n) = r(n)$, we assume that

$$\left.\frac{J_1(2\pi\sqrt{z})}{\pi\sqrt{z}}\right|_{z=0} = 1;$$

$S(n)$ is a singular series:

$$S(n) = \sum_{q=1}^{\infty} q^{-4} \sum_{\substack{p=0 \\ (p,q)=1}}^{q-1} S_{p,q}^4 \exp\left[-2\pi i \frac{p}{q} n\right],$$

$$S_{p,q} = \sum_{k=0}^{q-1} \exp\left[2\pi i \frac{p}{q} k^2\right].$$

2°. It is enough to consider only the case when the region Ω lies entirely in the first quadrant and its boundary has no common points, except the origin, with the plane $x_4 = 0$. The general case is reduced to this one by decomposing Ω into parts.

3°. Let $\mathfrak{E}$ be the projection onto the plane $x^4 = 0$ of the intersection of the conical region Ω with the unit sphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1.$$

$\mathfrak{E}$ is a three-dimensional quadrable set lying in the cube $\mathfrak{B}$:

$$0 \le x_1 \le 1, \qquad 0 \le x_2 \le 1, \qquad 0 \le x_3 \le 1.$$

Let $\delta > 0$ be a small real number, which we shall choose later; $\mathfrak{E}_\delta$ is the δ-neighborhood of the set $\mathfrak{E}$ (considered modulo 1, so that $\mathfrak{E}_\delta$ belongs to the cube $\mathfrak{B}$). Denoting by Ω_δ the conical region corresponding to $\mathfrak{E}$ (in the sense of projection), with solid angle ω_δ, we shall have

$$\omega_\delta = \omega + O(\delta). \tag{2}$$

Having fixed an integer $r \ge 1$, one can choose (cf. [(6)], Chap. I, Lemma 12) a function $\psi(x_1, x_2, x_3)$, defined on $\mathfrak{B}$, such that:

a) $\psi(x_1, x_2, x_3) = 0$, if $(x_1, x_2, x_3) \notin \mathfrak{E}_\delta$; $\psi(x_1, x_2, x_3) = 1$, if $(x_1, x_2, x_3) \in \mathfrak{E}$; $0 \le \psi(x_1, x_2, x_3) \le 1$, if $(x_1, x_2, x_3) \in \mathfrak{E}_\delta$;

b) $\psi(x_1, x_2, x_3)$ is expanded in a Fourier series:

$$\psi(x_1, x_2, x_3) = \sum_{l_1, l_2, l_3 = -\infty}^{+\infty} c_{l_1 l_2 l_3} \exp\left[2\pi i(l_1 x_1 + l_2 x_2 + l_3 x_3)\right],$$

$$c_{l_1 l_2 l_3} = \int_0^1 \int_0^1 \int_0^1 \psi(x_1, x_2, x_3) \exp\left[-2\pi i(l_1 x_1 + l_2 x_2 + l_3 x_3)\right] dx_1 dx_2 dx_3,$$

and moreover

$$|c_{l_1 l_2 l_3}| \ll \frac{1}{(\lambda_1 \lambda_2 \lambda_3)^r}, \tag{3}$$

where

$$\lambda_i = \begin{cases} \frac{1}{r}\delta |l_i|, & \text{if } l_i \ne 0, \\ 1, & \text{if } l_i = 0. \end{cases}$$

4°. Let n_1 be a positive integer, $n \leq n_1$. Put $\delta = n_1^{-1/18}$, $r = 17$, and denote

$$r(\omega, n, n_1) = \sum_{l_1, l_2, l_3 = -\infty}^{+\infty} c_{l_1 l_2 l_3}\, r_{l_1 l_2 l_3}(n).$$

In view of (1) and (3), after simple calculations we shall have

$$r(\omega, n, n_1) = \pi^2 n S(n) R(n_1) + O\left(n_1^{17/18+\varepsilon}\right), \tag{4}$$

where

$$R(n_1) = \sum_{l_1, l_2, l_3 = -\infty}^{+\infty} c_{l_1 l_2 l_3} \frac{J_1\left(2\pi\sqrt{l_1^2 + l_2^2 + l_3^2}\right)}{\pi\sqrt{l_1^2 + l_2^2 + l_3^2}}.$$

To compute $R(n_1)$ we use the method of I. M. Vinogradov ((6), Ch. III). Let $v(n)$ and $v(\Omega, n)$ be, respectively, the number of all integer points of the ball $x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq n$ and the number of integer points of this ball lying in the cone Ω. The following formulas are known:

$$v(n_1) = \frac{\pi^2}{2} n_1^2 + O\left(n_1^{3/2}\right), \qquad v(\Omega, n_1) = \frac{\omega}{2\pi^2} \frac{\pi^2}{2} n_1^2 + O\left(n_1^{3/2}\right).$$

Using the first of them and (4), we obtain

$$\sum_{n=1}^{n_1} \pi^2 n S(n) = \frac{\pi^2}{2} n_1^2 + O\left(n_1^{1+17/18+\varepsilon}\right). \tag{5}$$

Using the second and taking into account (5) and (2), we obtain

$$R(n_1) = \frac{\omega}{2\pi^2} + O\left(n_1^{-1/18+\varepsilon}\right).$$

Putting $n_1 = n$, we shall have

$$r(\omega, n, n) = \frac{\omega}{2\pi^2} \pi^2 n S(n) + O\left(n^{17/18+\varepsilon}\right).$$

5°. But $r(\omega, n) \leq r(\omega, n, n)$. Therefore

$$r(\omega, n) \leq \frac{\omega}{2\pi^2} \pi^2 n S(n) + O\left(n^{17/18+\varepsilon}\right).$$

In an entirely analogous way one can prove the same estimate from below. Therefore

$$r(\omega, n) = \frac{\omega}{2\pi^2}\pi^2 n S(n) + O\left(n^{17/18+\varepsilon}\right).$$

For odd n

$$S(n) > c.$$

Hence Theorem 1 follows.

Some generalization of Theorem 1 can also be proved in the same way:

Theorem 2. Let Ω be a four-dimensional convex conical region with vertex at the origin and solid angle $\omega > 0$; let n be a positive odd integer; let m, a_1, a_2, a_3, a_4 be integers satisfying the condition $n \equiv a_1^2 + a_2^2 + a_3^2 + a_4^2 \pmod{m}$. Denote by $r(\omega, n; m)$ the number of integer points of the sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = n$ lying in Ω and congruent to (a_1, a_2, a_3, a_4) modulo m. Then, as $n \to \infty$ and ω and m are fixed,

$$r(\omega, n; m) = \frac{1}{\rho(n, m)} \frac{\omega}{2\pi^2} r(n) \left(1 + O\left(n^{-1/18+\varepsilon}\right)\right),$$

where $\rho(n, m)$ is the number of solutions of the congruence $x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv n \pmod{m}$.

An analogous result also holds for the number $r_0(\omega, n; m)$ of integral primitive points of the sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = n$, lying in the region Ω and congruent to (a_1, a_2, a_3, a_4) modulo m.

Theorem 3. Under the hypotheses of Theorem 2, with $\gcd(a_1, a_2, a_3, a_4, m) = 1$,

$$r_0(\omega, n; m) = \frac{1}{\rho_0(n, m)} \frac{\omega}{2\pi^2} r_0(n) \left(1 + O\left(n^{-1/18+\varepsilon}\right)\right),$$

where $r_0(n)$ is the number of all integral primitive points of the sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = n$; $\rho_0(n, m)$ is the number of primitive $\pmod{m}$ solutions of the congruence $x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv n \pmod{m}$.

From Theorem 3 one can derive the following proposition from the arithmetic of quaternions, which has applications to the theory of ternary quadratic forms:

Theorem 4. Let Ω be a four-dimensional convex conical region with vertex at the origin and solid angle $\omega > 0$; let u be a positive odd number; let A and B be primitive quaternions satisfying $N(A)N(B) \mid u$. Denote by $\sigma(\omega, u; A, B)$ the number of primitive primary quaternions of norm u, lying in the region Ω and divisible by A on the left and by B on the right. Then, as $u \to \infty$ and for fixed ω, A, and B,

$$\sigma(\omega, u; A, B) = \frac{u}{N(A)N(B)} \frac{\omega}{2\pi^2} \frac{\prod_{p|u}\left(1+\frac{1}{p}\right)}{\prod_{p|N(A)}\left(1+\frac{1}{p}\right)\prod_{p|N(B)}\left(1+\frac{1}{p}\right)} \left(1 + O\left(u^{-1/18+\varepsilon}\right)\right),$$

where the product is taken over all primes p subject to the indicated conditions.

By an analogous method one may also consider the asymptotic distribution of integral points on ellipsoids $f(x_1, x_2, \ldots, x_s) = n$, where f is an arbitrary integral positive quadratic form with number of variables $s \geq 4$.

I express my deep gratitude to Prof. Yu. V. Linnik for valuable advice in carrying out this work.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
3 XII 1956

CITED LITERATURE

1. B. A. Venkov, *Elementary Number Theory*, Moscow-Leningrad, 1937.
2. N. Kloosterman, *Acta Math.*, **49**, 407 (1926).
3. Yu. V. Linnik, A. V. Malyshev, *DAN*, **89**, 209 (1953).
4. Yu. V. Linnik, *DAN*, **96**, 909 (1954).
5. E. M. Wright, *Quart. J. Math.*, **7**, 230 (1936).
6. I. M. Vinogradov, *The Method of Trigonometrical Sums in the Theory of Numbers*, Moscow, 1947.

Note: Figure translations are in progress. See original paper for figures.