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Abstract

Full Text

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MATHEMATICS

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A LOCAL LIMIT THEOREM FOR NONHOMOGENEOUS MARKOV CHAINS WITH A COUNTABLE NUMBER OF POSSIBLE STATES

(Presented by Academician A. N. Kolmogorov on 14 III 1957)

Consider a sequence

$$X_1, X_2, \dots, X_n, \dots$$

of random variables taking integer values and connected into a nonhomogeneous Markov chain with transition probabilities $p_{ij}^{(k)} = P\{X_k = j \mid X_{k-1} = i\}$ and absolute probabilities $p_{k|j} = P\{X_k = j\}$. Let

$$S_n = X_1 + X_2 + \dots + X_n.$$

We introduce the following conditions:

I. There exists a value j_0 such that $p_{ij_0}^{(k)} \geq \alpha > 0$ for all i and k.

II. $B_n^2 = DS_n \geq cn;\ c > 0$ *.

III. There exist uniformly bounded absolute moments $M|X_k|^s$ up to order s $(s \geq 3)$, inclusive.

IV. The greatest common divisor of all differences $j_l - j_0$ for which

$$\frac{1}{\ln n} \sum_{k=1}^{n} p_{k|j_l} \to \infty \qquad (n \to \infty),$$

is equal to one.

For the characteristic function $f_n(t)$ of the sum S_n the following lemmas hold.

Lemma 1 **. If conditions I—III are satisfied, then for $|t| \leq n^{1/6}$

$$f_n\left(\frac{t}{B_n}\right) = \exp\left[-\frac{t^2}{2} + it\frac{MS_n}{B_n}\right]\left(1 + \sum_{k=1}^{s-3}\frac{1}{n^{k/2}}P_k(it)\right) +$$

$$+\frac{1}{n^{(s-2)/2}}\left(|t|^s + |t|^{3(s-2)}\right)\exp\left[-\frac{t^2}{2}\right]O(1), \tag{1}$$

where $P_k(it)$ is a polynomial in it of degree $3k$. The coefficients of the polynomial are real, depend on n, but are uniformly bounded in n.

Here and below, the constant in the symbol $O(1)$ depends only on s.

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* Condition II is satisfied if, for example, $DX_k^\tau \geq c_1 > 0$ ([1]).

** Lemma 1 is valid if condition I is replaced by the condition that the coefficient of regularity $a^{(n)} \geq \delta > 0$, and the quantities X_k may be arbitrary $\mathfrak{A}_k$-measurable functions (for the definition of $a^{(n)}$ and $\mathfrak{A}_k$, see ([1])).

Lemma 2. If conditions I, IV are satisfied, then

$$\int_{n^{1/s}<|t|<\pi B_n}\left|f_n\left(\frac{t}{B_n}\right)\right| dt = O\left(\frac{1}{n^{(s-2)/2}}\right). \tag{2}$$

From the equality

$$B_nP\{S_n = m\} = \frac{1}{2\pi}\int_{|t|<n^{1/s}} f_n\left(\frac{t}{B_n}\right)\exp\left[-it\frac{m}{B_n}\right] dt+$$

$$+\frac{1}{2\pi}\int_{n^{1/s}<|t|<\pi B_n} f_n\left(\frac{t}{B_n}\right)\exp\left[-it\frac{m}{B_n}\right] dt$$

and the estimates (1), (2), the following theorem easily follows (see, for example, (2), § 51).

Theorem. Under conditions I-IV the expansion

$$B_nP\{S_n = m\} = g(x_{nm}) + \sum_{k=1}^{s-3}\frac{1}{n^{k/2}}P_k\left(-\frac{d}{dx_{nm}}\right)g(x_{nm}) + O\left(\frac{1}{n^{(s-2)/2}}\right),$$

holds, where

$$g(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right]; \qquad x_{nm} = \frac{m - MS_n}{B_n};$$

$P_k\left(-\frac{d}{dx}\right) g(x)$ means that in the polynomial $P_k(it)$ the powers $(it)^\nu$ are replaced by the expressions

$$(-1)^\nu \frac{d^\nu}{dx^\nu} g(x).$$

Remark. The relation

$$B_{nP}\{S_n = m\} - g(x_{nm}) \to 0 \qquad (n \to \infty)$$

holds uniformly in all m also in the case when condition III is replaced by the weaker condition:

IIIa. There exist uniformly bounded $M|X_k|^{2+\delta}$, $\delta > 0$.

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CITED LITERATURE

1. R. L. Dobrushin, *Probability Theory and Its Applications*, No. 1 (1956).

2. B. V. Gnedenko, A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, 1949.

Note: Figure translations are in progress. See original paper for figures.