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Abstract

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SOLUTION IN INTEGRAL FORM OF A MIXED PROBLEM FOR A SYSTEM OF TWO DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE

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Let us consider the system of equations, encountered in the theory of heat and mass transfer (^{1}),

$$\begin{gathered}\frac{\partial U}{\partial t} = a\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + b\frac{\partial V}{\partial t},\\ \frac{\partial V}{\partial t} = a'\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) + b'\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right),\\ (a + a' + bb')^2 \neq 4aa',\end{gathered} \tag{1}$$

with the initial conditions

$$U(x, y, 0) = F_1(x, y), \qquad V(x, y, 0) = F_2(x, y) \tag{2}$$

and the boundary conditions

$$U'_x(0, y, t) = U'_y(x, 0, t) = V'_x(0, y, t) = V'_y(x, 0, t) = 0, \tag{3}$$

$$\begin{gathered}U'_x(c, y, t) = \psi_1(y, t), \qquad U'_y(x, d, t) = X_1(x, t) \quad (t \geq 0),\\ V'_x(c, y, t) = \psi_2(y, t), \qquad V'_y(x, d, t) = X_2(x, t).\end{gathered} \tag{4}$$

The coefficients a, b, a', b' are constants. To solve the posed problem, we shall use the double Fourier transform, taken over the domain $R(-c \leq x \leq c,\ -d \leq y \leq d)$,

$$\mathfrak{F}\{\Phi(x, y)\} = \iint\limits_R \exp\left(-im\frac{\pi}{c}\xi - in\frac{\pi}{d}\eta\right)\Phi(\xi, \eta)\, d\xi\, d\eta = \varphi(m, n), \tag{5}$$

by means of which to a real function $\Phi(x, y)$, integrable in R, there is put in correspondence a function $\varphi(m, n)$ of two integer arguments m and n. In analogy with the terminology of operational calculus, the function $\Phi(x, y)$ will be called the original, and the function $\varphi(m, n)$ its image. The conditions $(^2)$ are known under which one may pass from the image $\varphi(m, n)$ to the original $\Phi(x, y)$ in the form

$$\Phi(x, y) = \frac{1}{4cd} \sum_{m,n}^{-\infty,\infty} \exp\left(im\frac{\pi}{c}x + in\frac{\pi}{d}y\right) \varphi(m, n) = \mathfrak{F}^{-1}\{\varphi(m, n)\}. \tag{6}$$

Let the function $\Phi_1(x, y)$ be integrable in the rectangle $(-2c \le x \le 2c,\ -2d \le y \le 2d)$, and the function $\Phi_2(x, y)$ be integrable in R. Then by the convolution of these functions we mean the double integral

$$\Phi_1(x, y) * \Phi_2(x, y) = \iint\limits_R \Phi_1(x - \xi, y - \eta)\Phi_2(\xi, \eta)\, d\xi\, d\eta. \tag{7}$$

If the function $\Phi_1(x, y)$ is periodic in the first argument with period $2c$ and in the second with period $2d$, then the basic convolution law holds:

$$\mathfrak{F}\{\Phi_1 * \Phi_2\} = \mathfrak{F}\{\Phi_1\} \cdot \mathfrak{F}\{\Phi_2\}. \tag{8}$$

It will be expedient for us to continue the functions U and V into the intervals $(-c \le x \le 0)$ and $(-d \le y \le 0)$ in an even manner, and their partial derivatives U'_x, U'_y, V'_x, V'_y in an odd manner, which can be done by virtue of the conditions (3). Then, assuming that $\mathfrak{F}$ can be interchanged with $\dfrac{\partial}{\partial t}$, after applying the transform (5) to problem (1)—(4), we obtain a system of two ordinary differential equations

$$\frac{du(m, n, t)}{dt} = -a\delta u(m, n, t) + 2as_1(t) + b\frac{dv(m, n, t)}{dt},$$

$$\frac{dv(m, n, t)}{dt} = -a'\delta v(m, n, t) - b'\delta u(m, n, t) + 2a's_2(t) + 2b's_1(t),$$

$$\delta = \pi^2\left(\frac{m^2}{c^2} + \frac{n^2}{d^2}\right), \tag{9}$$

$$s_j(t) = (-1)^m \int_{-d}^{d} \exp\left(-in\frac{\pi}{d}\eta\right) \Psi_j(\eta, t)\, d\eta +$$

$$+(-1)^n \int_{-c}^{c} \exp\left(-im\frac{\pi}{c}\xi\right) \chi_j(\xi, t)\, d\xi.$$

Here we assume that the conditions (2), after application of the Fourier transform (5), pass into the conditions

$$u(m,n,0) = f_1(m,n), \qquad v(m,n,0) = f_2(m,n). \tag{10}$$

By the known rules of operational calculus (^{3}), problem (9), (10), by means of the Laplace–Carson integral, is reduced to the algebraic system

$$(p + a\delta)\,\overline{u}(m,n,p) - bp\,\overline{v}(m,n,p) = pf_1(m,n) - bpf_2(m,n) + 2a\overline{s}_1(p),$$

$$b'\delta\,\overline{u}(m,n,p) + (p + a'\delta)\,\overline{v}(m,n,p) = pf_2(m,n) + 2a'\overline{s}_2(p) + 2b'\overline{s}_1(p), \tag{11}$$

where, as usual, the bar over the functions denotes their Laplace–Carson transforms. After finding $\overline{u}$ and $\overline{v}$ from the latter and passing to originals with respect to p, we shall have

$$u(m,n,t) = (A_1 e^{-\alpha_1\delta t} + A_2 e^{-\alpha_2\delta t}) f_1(m,n) + (A_3 e^{-\alpha_1\delta t} + A_4 e^{-\alpha_2\delta t}) f_2(m,n) +$$

$$+ (A_5 e^{-\alpha_1\delta t} + A_6 e^{-\alpha_2\delta t}) * s_1(t) + (A_7 e^{-\alpha_1\delta t} + A_8 e^{-\alpha_2\delta t}) * s_2(t), \tag{12}$$

where

$$\alpha_{1,2} = \frac{1}{2}\left[a' + r \pm \sqrt{(a'+r)^2 - 4aa'}\right] \quad (r = a + bb'),$$

with the minus sign corresponding to the index 1, and the plus sign to the index 2.

For the coefficients A_i $(i = 1, 2, \ldots, 8)$ the following relations hold:

$$A_1 = \gamma(\alpha_1 - a'), \qquad A_2 = \gamma(a' - \alpha_2), \qquad A_3 = -A_4 = \frac{A_7}{2\alpha_1} = -\frac{A_8}{2\alpha_2} = \gamma a' b,$$

$$A_5 = 2\gamma(\alpha_1 r - aa'), \qquad A_6 = 2\gamma(aa' - \alpha_2 r) \quad \left(\gamma = \frac{1}{\alpha_1 - \alpha_2}\right). \tag{13}$$

The sign ${}^{t}{*}$ denotes taking the convolution with respect to t from zero to t. The function $v(m,n,t)$ is expressed by means of the right-hand side of equality (12), where, instead of the coefficients A_i, one must put the coefficients B_i:

$$B_1 = -B_2 = \frac{B_5}{2a_1} = -\frac{B_6}{2a_2} = \gamma b', \qquad B_3 = \gamma(\alpha_1 - r), \qquad B_4 = \gamma(r - \alpha_2), \quad (14)$$

$$B_7 = 2\gamma a'(\alpha_1 - a), \qquad B_8 = 2\gamma a'(a - \alpha_2).$$

We now proceed to finding the originals with respect to m and n in equality (12). First note that, taking into account the known definition of the theta function (4):

$$\vartheta_3(v,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n} \qquad (z = e^{i\pi v},\ q = e^{i\pi\tau}),$$

it is easy to find, with the aid of (6), that the image $k(m,n,\zeta) = e^{-\delta\zeta}$ corresponds to the original

$$K(x,y,\zeta) = \frac{1}{4cd}\,\vartheta_3\left[\frac{x}{2c}, \frac{i\pi\zeta}{c^2}\right]\vartheta_3\left[\frac{y}{2d}, \frac{i\pi\zeta}{d^2}\right],$$

which, by virtue of the known properties of theta functions, is a periodic function in x of period $2c$ and in y of period $2d$. Taking the latter into account, it is not difficult to obtain the originals in the first two terms of equality (12) with the aid of relation (8), and in the remaining terms–with the aid of (6). Then, after obvious transformations, we shall have

$$U(x,y,t) = A_1J_{11} + A_2J_{21} + A_3J_{12} + A_4J_{22} + A_5J'_{11} + A_6J'_{21} + A_7J'_{12} + A_8J'_{22}, \quad (15)$$

where

$$J_{kl} = \frac{1}{4cd}\iint\limits_{R} \vartheta_3\left[\frac{x-\xi}{2c}, -\frac{i\pi\alpha_k t}{c^2}\right]\vartheta_3\left[\frac{y-\eta}{2d}, -\frac{i\pi\alpha_k t}{d^2}\right] F_l(\xi,\eta)\,d\xi\,d\eta,$$

$$J'_{kl} = \frac{1}{4cd}\int_0^t d\tau \left\{ \vartheta_3\left[\frac{c-x}{2c}, -\frac{i\pi\alpha_k(t-\tau)}{c^2}\right]\int_{-d}^{d}\vartheta_3\left[\frac{y-\eta}{2d}, -\frac{i\pi\alpha_k(t-\tau)}{d^2}\right]\Psi_l(\eta,\tau)\,d\eta \right.$$
$$\left. + \vartheta_3\left[\frac{d-y}{2d}, -\frac{i\pi\alpha_k(t-\tau)}{d^2}\right]\int_{-c}^{c}\vartheta_3\left[\frac{x-\xi}{2c}, -\frac{i\pi\alpha_k(t-\tau)}{c^2}\right] X_l(\xi,\tau)\,d\xi \right\}.$$

By virtue of the remark mentioned with respect to the function $v(m, n, t)$, its original $V(x, y, t)$ is expressed by means of the right-hand side of equality (15), in which only the coefficients A_i $(i = 1, 2, \dots, 8)$ must be replaced by the coefficients B_i from (14).

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Note: Figure translations are in progress. See original paper for figures.

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