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Abstract

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ON EXACT SOLUTIONS OF THE LINEARIZED PROBLEM OF A POINT EXPLOSION WITH BACK PRESSURE

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Let us consider the problem of a point explosion in a perfect gas in the formulation of L. I. Sedov (2), when the initial pressure p_1 is constant, while the density of the undisturbed medium varies according to the law $\rho_1 = Ar^{\omega}$, where A and ω are constants.

The self-similar problem with zero back pressure for adiabatic gas motions was solved in closed finite form (2) for arbitrary γ and ω in the cases of spherical, cylindrical, and plane symmetry. The solution becomes especially simple (2) if ω and γ are related by

$$\omega = \frac{7-\gamma}{\gamma+1}. \tag{1}$$

For what follows we introduce the system of dimensionless variables

$$\frac{v}{c} = f(\lambda, q), \qquad \frac{\rho}{\rho_2} = g(\lambda, q), \qquad \frac{p}{p_2} = h(\lambda, q),$$

$$\lambda = \frac{r}{r_2}, \qquad q = \frac{\gamma p_1}{\rho_1 c^2}. \tag{2}$$

Here v is the velocity; ρ is the density; p is the pressure; c is the velocity of the shock wave; r_2 is the radius of the shock wave; ρ_2 and p_2 are the density and pressure immediately behind the front of the shock wave. If the back pressure is not neglected, then the parameter q is essential, and the problem is not self-similar. In the dimensionless variables (2), the system of equations of one-dimensional unsteady gas motions with spherical symmetry can be represented in the form

$$(f-\lambda)\frac{\partial f}{\partial \lambda} + \frac{[2\gamma-(\gamma-1)q]\,[\gamma-1+2q]}{\gamma(\gamma+1)^2 g}\frac{\partial h}{\partial \lambda} + r_2\frac{dq}{dr_2}\frac{\partial f}{\partial q} - \frac{r_2}{2q}\frac{dq}{dr_2}f = 0;$$

$$(f-\lambda)\frac{\partial g}{\partial \lambda}+g\frac{\partial f}{\partial \lambda}+r_2\frac{dq}{dr_2}\frac{\partial g}{\partial q}+\left[\frac{2f}{\lambda}-\omega-\frac{2r_2}{\gamma-1+2q}\frac{dq}{dr_2}\right]g=0; \qquad (3)$$

$$(f-\lambda)\frac{\partial g}{\partial \lambda}+\gamma h\frac{\partial f}{\partial \lambda}+r_2\frac{dq}{dr_2}\frac{\partial h}{\partial q}+\left[\frac{2f}{\lambda}-\frac{2r_2\frac{dq}{dr_2}}{q(2\gamma-(\gamma-1)q)}\right]\gamma h=0.$$

In system (3), the first equation is obtained from the momentum equation, the second from the continuity equation, and the third equation is a consequence of the continuity equation and the condition of adiabaticity of the flow behind the shock front. A system of equations analogous to (3) was first used by N. S. Melnikova-Burnova ([2,3]) and A. Sakurai ([4]) in solving the linearized problem of an explosion in a medium of constant density.

The boundary conditions at the shock-wave front have the form

$$f(1,q)=\frac{2}{\gamma+1}(1-q), \qquad g(1,q)=1, \qquad h(1,q)=1. \qquad (4)$$

At the center of the explosion we have the condition

$$f(0,q)=0. \qquad (5)$$

Moreover, using the known solution ([2]) of the self-similar problem, we have the initial conditions at $q=0$:

$$f_0=f(\lambda,0)=\frac{2}{\gamma+1}\lambda, \qquad g_0=g(\lambda,0)=\lambda, \qquad h_0=h(\lambda,0)=\lambda^3. \qquad (6)$$

To determine the functions $f(\lambda,q)$, $g(\lambda,q)$, $h(\lambda,q)$, $r_2(q)$, one must find the solution of system (3) with conditions (4), (5), (6).

For small values of q, i.e., for times when the explosion is still sufficiently strong, the solution of the problem posed above may be sought in the form

$$\begin{aligned} &f(\lambda,q)=f_0(\lambda)+qf_1(\lambda)+\dots; &&g(\lambda,q)=g_0(\lambda)+qg_1(\lambda)+\dots;\\ &h(\lambda,q)=h_0(\lambda)+qh_1(\lambda)+\dots; &&\frac{r_2}{q}\frac{dq}{dr_2}=3(1+a_1q+\dots). \end{aligned} \qquad (7)$$

We shall consider the linearized problem, i.e., neglect terms of order q^2 and higher. Carrying out the linearization of equations (3), to determine $f_1(\lambda)$, $g_1(\lambda)$, $h_1(\lambda)$ and the constant a_1, we obtain the system of linear differential equations

$$g_0(f_0-\lambda)f_1' + \frac{2(\gamma-1)}{(\gamma+1)^2}h_1' + g_0\left(f_0' + \frac{3}{2}\right)f_1 + \left[-\frac{3}{2}f_0 + (f_0-\lambda)f_0'\right]g_1 - \frac{3a_1}{2}f_0 g_0 + \frac{4\gamma-(\gamma-1)^2}{\gamma(\gamma+1)^2}h_0' = 0;$$

$$g_0 f_1' + (f_0-\lambda)g_1' + \left(g_0' + \frac{2g_0}{\lambda}\right)f_1 + \left[f_0' + \frac{2f_0}{\lambda} + 3 - \omega\right]g_1 - \frac{6}{\gamma-1}g_0 = 0;$$

$$\gamma h_0 f_1' + (f_0-\lambda)h_1' + \left(h_0' + \frac{2\gamma h_0}{\lambda}\right)f_1 + \gamma\left[f_0' + \frac{2f_0}{\lambda}\right]h_1 - 3\left(a_1 + \frac{\gamma-1}{2\gamma}\right)h_0 = 0. \quad (8)$$

Primes in equations (8) denote differentiation with respect to λ. Taking into account (6) and (7), from the boundary conditions (4) and (5) we have:

$$f_1(1) = -\frac{2}{\gamma+1}, \qquad g_1(1) = h_1(1) = 0, \qquad f_1(0) = 0. \quad (9)$$

Thus, in order to find the functions f_1, g_1, h_1 and the constant a_1, it is necessary to solve system (8) with conditions (9). The general solution of system (8) has the form

$$f_1 = \frac{1-\gamma}{\gamma+1}\left(\alpha_1\lambda + c_2\lambda^{r_2+1} + c_3\lambda^{r_3+1}\right);$$

$$g_1 = \alpha_2\lambda + c_1\lambda^{r_1+1} - \frac{(r_2+4)(1-\gamma)}{(r_2+3)(1-\gamma)+6\gamma}c_2\lambda^{r_2+1} + k_1 c_3\lambda^{r_3+1}; \quad (10)$$

$$h_1 = \alpha_3\lambda^3 + \frac{5\gamma+1}{(2r_1+6)(\gamma-1)}c_1\lambda^{r_1+3} - \frac{(\gamma r_2+3+3\gamma)(1-\gamma)}{(r_2+3)(1-\gamma)+6\gamma}c_2\lambda^{r_2+3} + k_2 c_3\lambda^{r_3+3}.$$

Here c_1, c_2, c_3 are arbitrary constants; k_1, k_2 are certain known constants. The constants $\alpha_1, \alpha_2, \alpha_3$ are expressed in terms of a_1 by the formulas

$$\alpha_1 = \frac{36\left(a_1 + \frac{\gamma-1}{2\gamma}\right) - \frac{12(5\gamma+1)}{(\gamma-1)^2} - \frac{18\left[4\gamma-(\gamma-1)^2\right] - 18a_1\gamma(\gamma+1)}{\gamma(1-\gamma)}}{7\gamma+71}; \quad (11)$$

$$\alpha_2 = \frac{4}{3}\frac{\gamma-1}{\gamma+1}\alpha_1 + \frac{2}{\gamma-1}; \qquad \alpha_3 = (\gamma-1)\alpha_1 + \left(a_1 + \frac{\gamma-1}{2\gamma}\right).$$

Fig. 1

Figure 1: Fig. 1

The constants r_1, r_2, r_3 are functions of the parameter γ and are determined by the formulas

$$r_1 = \frac{3(\gamma+1)}{\gamma-1}, \qquad r_{2,3} = \frac{-\delta_1 \mp \sqrt{\delta_1^2 - 8(\gamma^2-1)\delta_2}}{4(\gamma^2-1)},$$

where

$$\delta_1 = 31\gamma^2 - 14\gamma - 29, \qquad \delta_2 = 33\gamma^2 - 70\gamma - 71.$$

Since r_3, for any γ, is a negative quantity whose absolute value is greater than 1, in order to satisfy the boundary condition at the center $f(0) = 0$ one must set $c_3 = 0$.

To find the constants c_1 and c_2 and the constant a_1 entering into the definition of the dependence $r_2(q)$, we use the conditions at the shock wave (9). Setting $c_3 = 0$ and $\lambda = 1$ in the solution (10) and taking (9) into account, to determine c_1, c_2, a_1 we obtain a system of inhomogeneous linear equations whose coefficients depend on γ:

Fig. 1

$$c_2 + \alpha_1 - \frac{2}{\gamma-1} = 0;$$

$$c_1 + \frac{4}{3}\frac{\gamma-1}{\gamma+1}\alpha_1 - \frac{(r_2+4)(1-\gamma)\left(\frac{2}{\gamma-1} - \alpha_1\right)}{(r_2+3)(1-\gamma)+6\gamma} + \frac{2}{\gamma-1} = 0;$$

$$\frac{5\gamma+1}{(2r_1+6)(\gamma-1)}c_1 + (\gamma-1)\alpha_1 + a_1 \tag{12}$$

$$-\frac{(\gamma r_2 + 3 + 3\gamma)(1-\gamma)}{(r_2+3)(1-\gamma)+6\gamma}\left(\frac{2}{\gamma-1} - \alpha_1\right) + \frac{\gamma-1}{2\gamma} = 0,$$

where it is necessary to take into account the relation between α_1 and a_1, which is known according to (11).

Solving the systems (11), (12), one can find the dependence of $\alpha_1, \alpha_2, \alpha_3, c_1, c_2, a_1$ on γ. The results of calculations* for some values of γ, and hence also ω, are given in Table 1.

Table 1

Fig. 2

Figure 2: Fig. 2

γ	α_1	α_2	α_3	c_1	c_2	a_1
1.2	−9.767	8.816	−7.824	−23.44	19.77	−5.954
1.4	−2.678	4.405	−3.756	−9.578	7.678	−2.828
5/3	−0.909	2.697	−2.318	−5.229	3.909	−1.912
3.0	−0.130	0.913	−1.465	−1.643	1.130	−1.538
7.0	−0.122	0.212	−1.521	−0.514	0.455	−1.218

After the calculation of the indicated constants, the problem of finding the functions $f_2(\lambda), g_1(\lambda), h_1(\lambda)$ is completely solved. The graphs of $g_1(\lambda)$, $\overline{f}_1(\lambda) = -\dfrac{\gamma+1}{2} f_1(\lambda)$ for various γ are given in Figs. 1 and 2.

* R. I. Bormotova took part in the calculations.

We note that in the case of plane and cylindrical shock waves there also exist exact solutions of type (6) for the self-similar problem. In this case

Fig. 2

the values of ω are: $\omega = 1$ (plane case), $\omega = \dfrac{4}{\gamma+1}$ (cylindrical case). For these values of ω, in a manner similar to that set out above, one can also obtain an exact solution of the linearized problem with allowance for back pressure.

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Note: Figure translations are in progress. See original paper for figures.