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Abstract

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ON THE THEORY OF RADICALS OF ASSOCIATIVE RINGS

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In the works of A. G. Kurosh and Amitsur ([1–3]) an axiomatic approach to the concept of the radical of a ring was found; this laid the foundation for the general theory of radicals. The present work is devoted to the further construction of the general theory of radicals, primarily for the class of associative rings.

Consider a radical* R satisfying the condition

I. Every ideal of an R-radical ring is an R-radical ring.

As usual, we shall call a ring K subdirectly irreducible if the intersection of all its nonzero ideals is a nonzero ideal—the heart of K.

It is easy to show that if the radical R satisfies condition I, then the class of all subdirectly irreducible rings with R-radical heart defines an upper radical in the sense of Kurosh (for the definition of an upper radical see ([1])).

Let R be a given radical. We shall call a radical Q complementary to R if it is the largest among all radicals having, in any ring K, zero intersection with R (in the case when such a radical exists). The radical complementary to R will be denoted by R'.

Theorem 1. *If the radical R satisfies condition I, then there exists a radical R' complementary to it; moreover, R' is the upper radical defined by the class of all subdirectly irreducible rings with R-radical heart. The R'-radical rings are precisely the strongly R-semisimple rings.***

A radical R satisfying condition I will be called a **supernilpotent radical** if in every ring K its R-radical contains all nilpotent ideals of K.

A radical R satisfying condition I will be called a **subidempotent radical** if every ideal of an R-radical ring is idempotent.

Recall that a ring K is called **prime** ([4]) if from $AB = 0$, where A and B are ideals in K, it follows that at least one of the equalities $A = 0$ or $B = 0$ holds. As usual, by the annihilator A^* of an ideal A in an arbitrary ring K we shall mean the ideal consisting of all such elements x of the ring K that $Ax = xA = 0$. As usual, we shall regard as algebraic properties only those which are not changed under isomorphic mappings.

Let us now consider the class of associative rings possessing some algebraic property σ, or, more briefly, σ-rings. We shall call an ideal P of an arbitrary ring K a σ-ideal if the factor ring $\overline{K} = K/P$ is a nonzero σ-ring.

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* Here and below, a radical is understood in the sense of the most general definition of this concept, given in ([1]).

** We shall call a ring K strongly R-semisimple if every homomorphic image of it is an R-semisimple ring. See also Definition 7.2 and Theorem 7.1 in ([2]).

We shall call a class of σ-rings a **special class of rings** if the property σ satisfies the following conditions:

II, 1. Every σ-ring is a prime ring.

II, 2. Every nonzero ideal of a σ-ring is a σ-ring.

II, 3. Every extension K of a nonzero σ-ring A is an extension of its annihilator A^* in K by means of some σ-ring.

Let B be an ideal in a ring A, and let A be an ideal in a ring K. Denote by $B : A$ the set of all such x in K that $Ax \subseteq B$, $xA \subseteq B$.

A class of σ-rings will be a special class of rings if and only if the following conditions are fulfilled:

III, 1. If a σ-ideal P of a ring K does not contain the ideal A, then

$$(P \cap A) : A = P.$$

III, 2. If a σ-ideal P of a ring K does not contain the ideal A, then $P \cap A$ is a σ-ideal in the ring A.

III, 3. If P_0 is a σ-ideal in the ring A, where A is an ideal of the ring K, then $P_0 : A$ is a σ-ideal in K, and moreover

$$P_0 = (P_0 : A) \cap A.$$

If a class of σ-rings is a special class of rings, then the corresponding σ-ideals shall be called **special ideals**.

Every special class of rings determines an upper radical; this radical we shall call a **special radical**. Specially radical rings are those rings which cannot be mapped homomorphically onto nonzero rings from the given special class of rings, i.e. rings without special ideals. Every special radical R is a supernilpotent radical, and R is the intersection of all special ideals of the ring. The question remains open whether every supernilpotent radical, in particular the locally nilpotent radical of Levitzki ([5]), is a special radical. Specially semisimple rings are subdirect sums of rings from the corresponding special class of rings.

Theorem 2. *The class of all subdirectly irreducible rings with an idempotent heart possessing an arbitrary but fixed algebraic property φ is a special class of rings.*

We shall call two radicals R and S **mutually complementary** if there exist R' and S', with $S = R'$ and $R = S'$. If R and S are mutually complementary, then $R = (R')' = R''$ and $S = (S')' = S''$.

We shall call a radical R **dual** if there exist R' and R'', with $R = R''$, i.e. if R and R' are mutually complementary.

Theorem 3. *If R is an arbitrary supernilpotent (subidempotent) radical, then there exist R' and R'', with R' and R'' mutually complementary; R' is a dual subidempotent radical, and R'' is a dual special radical. R'' is the least dual radical containing R.*

Let us note that if R is an arbitrary supernilpotent or subidempotent radical, then all subdirectly irreducible rings split into two classes: the class of subdirectly irreducible rings with R-radical heart, and the class of subdirectly irreducible rings with R-semisimple (R'-radical) heart; these two classes determine, respectively, the upper radicals R' and R''. Therefore a supernilpotent (subidempotent) radical R will be dual if and only if it is the upper radical determined by the class of all subdirectly irreducible rings with R-semisimple heart.

If R is an arbitrary supernilpotent (subidempotent) radical, then the subdirectly irreducible rings with R-semisimple (R-radical) heart are subdirectly irreducible rings with idempotent

core possessing the algebraic property φ, where φ is the property of a ring to be R-semisimple (R-radical).

The following duality theorem for radicals holds.

Theorem 4. *Let M_φ be the class of all subdirectly irreducible rings with idempotent core possessing an arbitrary but fixed algebraic property φ, and let $M_{\tilde{\varphi}}$ be the class of all other subdirectly irreducible rings. Then the classes M_φ and $M_{\tilde{\varphi}}$ define, respectively, the upper radicals R_φ and $R_{\tilde{\varphi}}$, and R_φ and $R_{\tilde{\varphi}}$ are mutually complementary; R_φ is a dual special radical, and $R_{\tilde{\varphi}}$ is a dual subidempotent radical. In the manner described, all dual supernilpotent and subidempotent radicals are obtained.*

Let us note that the radical $R_{\tilde{\varphi}}$ is the intersection of all such ideals of the ring whose quotient rings are subdirectly irreducible with R_φ-radical core.

Theorem 5. *Let R be a given supernilpotent radical and K a strongly R-semisimple ring. If the annihilators of all such ideals of K whose quotient rings are nonzero subdirectly irreducible rings are different from zero, then K is a discrete direct sum of simple rings belonging to a suitable special class of rings. Conversely, every discrete direct sum of simple rings belonging to a given special* *class of rings M is a strongly R-semisimple ring, where R is the special radical determined by the class M.*

Corollary. If, in a strongly R-semisimple ring K, where R is a special radical, the annihilator of every special ideal is different from zero, then K is a discrete direct sum of simple rings belonging to the corresponding special class of rings.

The classes of primary rings, primitive rings ([6]), and simple rings with identity are special classes of rings. The special radicals defined by them are, respectively, the radicals of Baer–McCoy ([7,4,8]), Jacobson ([6]), and Brown–McCoy and Segal ([9,10]). The Baer–McCoy radical R_m is not a dual special radical. By Theorem 3, R'_m and R''_m exist. R'_m is precisely Blair' s hereditary idempotent (f-regular) radical ([11]), while R''_m will be a new dual special radical. We shall call it the **antisimple radical** ([12]). The antisimple radical is the upper radical determined by the class of all subdirectly irreducible rings with idempotent core, while Blair' s radical is the upper radical determined by the class of all subdirectly irreducible rings with nilpotent core. The Baer–McCoy radical is the least supernilpotent radical, whereas the antisimple radical is the least dual supernilpotent radical. It remains an open question whether the Jacobson radical is a dual radical. The Brown–McCoy and Segal radical is a dual special radical.

It is easy to show that the class of all rings without zero divisors is a special class of rings. The special radical N determined by this class is the intersection of all completely prime** ideals of the ring and contains all its nil-elements. We shall call N-radical rings **generalized nil-rings**; in the commutative case these are precisely nil-rings. N-semisimple rings are subdirect sums of rings without zero divisors. Full matrix rings over bodies, as well as bodies, form special classes of rings. The field with two elements forms a special class of rings. Let us consider the dual special radical R_M determined by this class and the radical R'_M complementary to it. R'_M-ra-

* If K is a discrete direct sum of simple rings, then, obviously, the annihilator of any ideal $B \neq K$ is a nonzero ideal.

** An ideal P is called **completely prime** if from $ab \equiv 0(P)$ it follows that either $a \equiv 0(P)$ or $b \equiv 0(P)$.

radical rings are precisely Boolean rings, i.e., rings in which for every element a the equality $a^2 = a$ holds. Thus, in associative rings the **Boolean radical** is defined–this is the dual subidempotent radical.

In the recently published paper of Goldie (13), a number of results were obtained concerning the question of the uniqueness of the representation of Jacobson semisimple rings as a subdirect sum of primitive rings. Analogous results also hold for any special semisimple rings, since in the cited paper only the fact is used that the class of primitive rings is a special class of rings.

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Note: Figure translations are in progress. See original paper for figures.