A BOUNDARY-VALUE PROBLEM OF RIEMANN-PROBLEM TYPE FOR SYSTEMS OF FIRST-ORDER DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE

At the present time, on the basis of the system of differential equations

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

L. G. MIKHAILOV

A BOUNDARY-VALUE PROBLEM OF RIEMANN-PROBLEM TYPE FOR SYSTEMS OF FIRST-ORDER DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE

(Presented by Academician A. N. Kolmogorov, 14 IX 1956)

At the present time, on the basis of the system of differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= a(x,y)u + b(x,y)v + f(x,y), \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= c(x,y)u + d(x,y)v + g(x,y) \end{aligned} \tag{1}$$

a theory of functions has been constructed analogous to the theory of analytic functions. System (1) is essentially reduced to the equation

$$\frac{\partial U}{\partial \bar{z}} = A(z)\bar{U}, \qquad U = u + iv. \tag{2}$$

I. N. Vekua ([1,3]) gave the fundamental formula

$$U(z) = \varphi(z)e^{\omega(z)}, \tag{3}$$

establishing the connection of $U(z)$ with the analytic function $\varphi(z)$ (see also ([7])). With the aid of this formula, such theorems as the uniqueness theorem, the argument principle, Liouville' s theorem, etc., are carried over to solutions of system (1). For system (1), I. N. Vekua ([1]) also considered a boundary-value problem of the form $\alpha u + \beta v$, analogous to the Hilbert problem in the theory of analytic functions.*

In the present work a boundary-value problem is studied that is analogous to the second main boundary-value problem of the theory of analytic functions—the Riemann problem. More general boundary-value problems are also studied.

We shall consider equation (2) in the whole plane E, under the condition that the coefficient $A(z)$ is bounded in E, is continuous in E except for a finite number of rectifiable Jordan curves, and at infinity satisfies the condition

$|A(z)| \leq M/|z|^{\alpha},\ \alpha > 1$. In equation (2), and everywhere below, the derivative $\partial U/\partial \bar{z}$ is understood in the generalized sense of Pompeiu ([1]).

Formula (3) admits an inversion ([3]). The inversion formula may be written in the form

$$U(z) = \varphi(z)\left[1 + \iint_E \Gamma_1^{\varphi}(z,\zeta)\,dT + \iint_E \Gamma_2^{\varphi}(z,\zeta)\,dT\right], \tag{4}$$

where $\Gamma_1^{\varphi}, \Gamma_2^{\varphi}$ are the resolvents of a certain integral equation (cf. ([3])). Formulas (3) and (4) establish a one-to-one correspondence between regular ([1]) solutions of equation (2) and analytic functions. This fact is used extensively below. Taking for $\varphi(z)$ in formula (4) the powers z^k and iz^k, we obtain analogues of the powers $U_{2k}(z)$ and $U_{2k+1}(z)$. We construct

* Here we follow the terminology of F. D. Gakhov ([5]).

then the analogue of a polynomial is

$$U_P(z) = \sum_{k=0}^{2n+1} A_k U_k(z),$$

where A_k are real constants.

Analogue of the generalized Liouville theorem. *If $U(z)$ is a regular solution of equation (2), continuous in the whole plane and having finite order at infinity, then $U(z) \equiv U_P(z)$.*

Let L be a set of a finite number of smooth Jordan lines.

Definition. A solution of equation (2) is called a piecewise-regular solution if it is regular everywhere outside L and can be continuously continued[(6)] to L from the left and from the right except possibly at the ends, in whose vicinity it admits the estimate $|U(z)| \leq K/|z-c|^{\lambda}$, $\lambda < 1$.

We shall further restrict ourselves to the case when L consists of closed contours, and let S^+ and S^- denote the same as in the usual Riemann problem[(6)].

Formulation of the Riemann problem. Find a piecewise-regular solution of equation (2), having finite order at infinity, satisfying the boundary condition on the contour

$$U^+(t) = G(t)U^-(t) + g(t), \tag{5}$$

where $G(t)$ and $g(t)$ are prescribed functions of points of the contour satisfying the Hölder condition, and $G(t) \neq 0$.

The method for solving the problem is similar to the method of F. D. Gakhov[(5)] for solving the usual Riemann problem. It consists in successively simplifying the boundary condition with the aid of the ordinary canonical function $\chi(z)^{(6)}$. The homogeneous boundary condition (5) is reduced to the condition of continuity on the contour

$$\frac{U^+(t)}{\chi^+(t)} = \frac{U^-(t)}{\chi^-(t)}.$$

Applying the analogue of the generalized Liouville theorem, we obtain the general solution of the homogeneous problem

$$U(z) = V_P(z)\chi(z), \tag{6}$$

where $V_P(z)$ is the analogue of a polynomial for the equation

$$\frac{\partial V}{\partial \bar{z}} = A(z)\frac{\overline{\chi(z)}}{\chi(z)}\overline{V}.$$

We note that the solution of the homogeneous problem can be obtained directly from formula (4). The nonhomogeneous boundary condition (5) is transformed into the problem of determining the piecewise-regular function $U(z)/\chi(z)$ from the jump

$$\frac{U^+(t)}{\chi^+(t)} - \frac{U^-(t)}{\chi^-(t)} = \frac{g(t)}{\chi^+(t)}.$$

The general solution of the nonhomogeneous problem is given by the formula

$$U(z) = \chi(z)[V_P(z) + W(z)], \tag{7}$$

where $W(z)$ is the analogue of an integral of Cauchy type.

Let $\operatorname{Ind}_L G(t) = \varkappa$. With respect to solutions vanishing at infinity, on the basis of formulas (6) and (7), the following theorem is proved.

Theorem. *The homogeneous Riemann problem for $\varkappa > 0$ has $2\varkappa$ linearly independent solutions (in the sense of combinations with real coefficients). For $\varkappa \leq 0$ it has no solutions. The nonhomogeneous problem for $\varkappa \geq 0$ is always solvable, while for $\varkappa < 0$ the necessary and sufficient conditions for solvability consist in the fulfillment of $(-\varkappa)$ equalities of the form*

$$\int_L \psi(t)t^k\,dt = 0,$$

$$k = 0, 1, \dots, -\varkappa - 1,$$

where $\psi(t)$ is a certain function expressible in terms of $G(t)$, $g(t)$, and $A(z)$.

The Riemann problem is solved analogously in the case of open contours or discontinuous coefficients. The Riemann-Hilbert problem

$$U^+[\alpha(t)] = G(t)U^-(t) + g(t)$$

is solved in approximately the same way as the usual Riemann-Hilbert problem(8). Let us consider two further problems.

Problem 1 with boundary condition

$$a(t)U^+(t) + \int_L A(t,\tau)\,U^+(\tau)\,d\tau - \left[b(t)U^-(t) + \int_L B(t,\tau)\,U^-(\tau)\,d\tau\right] = f(t).$$

Problem 2. Find a pair of functions: $U(z)$—a regular solution of equation (2) in S^+, and $\Phi(z)$, analytic in S^-, satisfying the boundary condition

$$a(t)U^+(t) + \int_L A(t,\tau)\,U^+(\tau)\,d\tau -$$

$$-\sum_{k=0}^{n}\left[b_k(t)\frac{d^k\Phi^-(t)}{dt^k} + \int_L B_k(t,\tau)\frac{d^k\Phi^-(\tau)}{d\tau^k}\,d\tau\right] = f(t).$$

Using the representation by a generalized Cauchy-type integral, Problems 1 and 2 can be reduced to equivalent singular integral equations (cf. (9)).

The author is deeply grateful to F. D. Gakhov for suggesting the topic and supervising the work, and also to I. N. Vekua for valuable comments during the publication of the paper.

Tajik State
University

Received
21 VII 1956

REFERENCES

(^{1}) I. N. Vekua, *Matem. sborn.*, **31** (73): 2 (1952).
(^{2}) I. N. Vekua, *DAN*, **89**, No. 5 (1953).
(^{3}) I. N. Vekua, *DAN*, **98**, No. 2 (1954).
(^{4}) I. N. Vekua, *DAN*, **100**, No. 2 (1955).
(^{5}) F. D. Gakhov, *Izv. Kaz. fiz.-matem. obshch.*, **14**, 3 (1949).
(^{6}) N. I. Muskhelishvili, *Singular Integral Equations*, Moscow-Leningrad, 1946.

(^{7}) L. Bers, *Theory of Pseudo-analytic Functions*, N. Y., 1953.
(^{8}) D. A. Kveselava, *Tr. Tbilissk. matem. inst.*, **16** (1948).
(^{9}) Yu. M. Krikunov, *DAN*, **85**, No. 2 (1952).

Note: Figure translations are in progress. See original paper for figures.