# APPROXIMATE COMPUTATION OF MULTIPLE INTEGRALS BY MEANS OF METHODS OF NUMBER THEORY

1957

SovietRxiv

## Abstract

## Full Text

## MATHEMATICS

**N. M. KOROBOV**

# APPROXIMATE COMPUTATION OF MULTIPLE INTEGRALS BY MEANS OF METHODS OF NUMBER THEORY

*(Presented by Academician I. M. Vinogradov on 21 VI 1957)*

Consider a function $f(x_1, \dots, x_s)$ having period equal to one in each of the variables $x_1, \dots, x_s$. We shall assume that $f(x_1, \dots, x_s)$ expands in the unit $s$-dimensional cube into an absolutely convergent Fourier series:

$$f(x_1, \dots, x_s) = \sum_{m_1, \dots, m_s = -\infty}^{\infty} C(m_1, \dots, m_s) \exp[2\pi i(m_1 x_1 + \dots + m_s x_s)]. \quad (1)$$

Let us denote by $\sigma$ the sum of the series formed from the absolute values of the Fourier coefficients:

$$\sigma = \sum_{m_1, \dots, m_s = -\infty}^{\infty} |C(m_1, \dots, m_s)|.$$

**Theorem 1.** Let $p > s$ be prime and $\xi_\nu(k) = \left\{ \frac{k^\nu}{p^2} \right\}$ $(\nu = 1, 2, \dots, s)^*$.

If in the unit $s$-dimensional cube the derivative

$$\frac{\partial^{2s} f(x_1, \dots, x_s)}{\partial x_1^2 \cdots \partial x_s^2}$$

is continuous and, for any integers $j_1 < j_2 < \dots < j_r$ $(1 \le r \le s,\ 1 \le j_1, j_r \le s)$, the derivatives

$$\frac{\partial^{2r} f(x_1, \dots, x_s)}{\partial x_{j_1}^2 \cdots \partial x_{j_r}^2}$$

are bounded in absolute value by a common constant $C$, then for $N = p^2$ the estimate

$$\left| \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_s)\, dx_1 \cdots dx_s - \frac{1}{N} \sum_{k=1}^{N} f[\xi_1(k), \dots, \xi_s(k)] \right| \leq$$

$$\leq \frac{(s-1)\sigma}{\sqrt{N}} + \frac{sC}{10N}.$$

The proof of the theorem is based on the following simple lemma.

**Lemma.** If at least one of the integers $m_1, \dots, m_s$ is not divisible by $p$, where $p > s$ is prime, then

$$\left| \sum_{k=1}^{p^2} \exp\left[2\pi i \frac{m_1 k + \dots + m_s k^s}{p^2}\right] \right| \leq (s-1)p.$$

**Proof.** Introduce the notation:

$$\varphi(k) = m_1 k + \dots + m_s k^s; \qquad S = \sum_{k=1}^{p^2} \exp\left[2\pi i \frac{\varphi(k)}{p^2}\right].$$

Splitting the summation interval of the sum $S$ into parts of length $p$ and using the obvious congruence

$$\varphi(x + py) \equiv \varphi(x) + \varphi'(x)py \pmod{p^2},$$

we obtain

$$S = \sum_{x=1}^{p} \sum_{y=0}^{p-1} \exp\left[2\pi i \, \frac{\varphi(x+py)}{p^2}\right] =$$

$$= \sum_{x=1}^{p} \exp\left[2\pi i \, \frac{\varphi(x)}{p^2}\right] \sum_{y=0}^{p-1} \exp\left[2\pi i \, \frac{\varphi'(x)y}{p}\right].$$

The degree of the polynomial $\varphi'(x)$ is less than $p$; its coefficients, by the conditions of the lemma, are not all divisible by $p$. Thus, denoting by $T$ the number of solutions of the congruence $\varphi'(x) \equiv 0 \pmod{p}$, we obtain:

---

* Here $\{A\}$ denotes the fractional part of the number $A$.

$$|S| \leq \sum_{x=1}^{p} \left| \sum_{y=0}^{p-1} \exp\left[2\pi i \frac{\varphi'(x)y}{p}\right] \right| = Tp \leq (s-1)p.$$

We now pass to the proof of the theorem. In view of (1) we have:

$$\frac{1}{N} \sum_{k=1}^{N} f[\xi_1(k), \dots, \xi_s(k)] = \tag{2}$$

$$= \frac{1}{N} \sum_{m_1,\dots,m_s=-\infty}^{\infty} C(m_1, \dots, m_s) \sum_{k=1}^{N} \exp\{2\pi i[m_1\xi_1(k) + \dots + m_s\xi_s(k)]\} =$$

$$= C(0, \dots, 0) + \frac{1}{N} \sum_{m_1,\dots,m_s=-\infty}^{\infty}{}' C(m_1, \dots, m_s) \sum_{k=1}^{p^2} \exp\left[2\pi i \frac{m_1k + \dots + m_sk^s}{p^2}\right]$$

(here $\sum'$ denotes summation over all systems $m_1, \dots, m_s$, except for the system $m_1 = \dots = m_s = 0$). Observing that

$$C(0, \dots, 0) = \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_s)\, dx_1 \cdots dx_s,$$

we obtain from (2):

$$\left| \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_s)\, dx_1 \cdots dx_s - \frac{1}{N} \sum_{k=1}^{N} f[\xi_1(k), \dots, \xi_s(k)] \right| =$$

$$= \frac{1}{N} \left| \sum_{m_1,\dots,m_s=-\infty}^{\infty}{}' C(m_1, \dots, m_s) \sum_{k=1}^{p^2} \exp\left[2\pi i \frac{m_1k + \dots + m_sk^s}{p^2}\right] \right| \leq$$

$$\leq \frac{1}{N} \sum_{m_1,\dots,m_s=-\infty}^{\infty}{}' |C(m_1, \dots, m_s)| \left| \sum_{k=1}^{p^2} \exp\left[2\pi i \frac{m_1k + \dots + m_sk^s}{p^2}\right] \right|. \tag{3}$$

Let $d$ denote the greatest common divisor of the quantities $m_1, \dots, m_s$, and split the systems $m_1, \dots, m_s$ into two classes: in the first class are the systems for which $d \equiv 0 \pmod{p}$, and in the second class all the remaining systems. Applying to the systems of the first class the trivial estimate

$$\left|\sum_{k=1}^{p^2} \exp\left[2\pi i\,\frac{m_1 k + \dots + m_s k^s}{p^2}\right]\right| \leq N$$

and, using for systems of the second class the estimate indicated in the lemma, we obtain:

$$\begin{aligned}
&\frac{1}{N} \sum_{m_1,\dots,m_s=-\infty}^{\infty}{}' |C(m_1,\dots,m_s)| \left|\sum_{k=1}^{p^2} \exp\left[2\pi i \frac{m_1 k + \dots + m_s k^s}{p^2}\right]\right| \\
&\leq \sum_{d\equiv 0 \pmod p}{}' |C(m_1,\dots,m_s)| + \frac{(s-1)p}{N} \sum_{d\not\equiv 0 \pmod p} |C(m_1,\dots,m_s)| \qquad (4) \\
&\leq \sum_{m_1,\dots,m_s=-\infty}^{\infty}{}' |C(m_1 p,\dots,m_{sp})| + \frac{(s-1)\sigma}{\sqrt{N}}.
\end{aligned}$$

Let $m_1,\dots,m_s$ be a system of integers not all equal to zero simultaneously. Suppose that $r$ of these numbers are different from zero ($1 \leq r \leq s$). Denote the moduli of the nonzero numbers by $n_1,\dots,n_r$. Using the equality

$$C(m_1,\dots,m_s) = \int_0^1 \cdots \int_0^1 f(x_1,\dots,x_s) \exp[-2\pi i(m_1 x_1 + \dots + m_s x_s)]\, dx_1 \cdots dx_s,$$

by virtue of the conditions of the theorem, after $2r$-fold integration by parts we obtain

$$|C(m_1,\dots,m_s)| \leq \frac{C}{(2\pi)^{2r}(n_1\cdots n_r)^2} = \frac{C}{(2\pi)^{2s}} \frac{(2\pi)^{2(\delta_{m_1}+\dots+\delta_{m_s})}}{\left[(|m_1|+\delta_{m_1})\cdots(|m_s|+\delta_{m_s})\right]^2}, \qquad (5)$$

where the quantities $\delta_{m_\nu}$, for $\nu = 1, 2, \dots, s$, are defined by the equalities

$$\delta_{m_\nu} = \begin{cases} 1, & \text{if } m_\nu = 0, \\ 0, & \text{if } m_\nu \neq 0. \end{cases}$$

But then

$$\sum_{m_1,\dots,m_s=-\infty}^{\infty}{}' |C(m_1 p,\dots,m_{sp})| \leq \frac{C}{(2\pi p)^{2s}} \sum_{m_1,\dots,m_s=-\infty}^{\infty}{}' \frac{(2\pi p)^{2(\delta_{m_1}+\dots+\delta_{m_s})}}{\left[(|m_1|+\delta_{m_1})\cdots(|m_s|+\delta_{m_s})\right]^2}$$

$$= C\left[\left(\frac{1}{4\pi^2 p^2}\sum_{m=-\infty}^{\infty}\frac{(2\pi p)^{2\delta_m}}{(|m|+\delta_m)^2}\right)^s - 1\right] = C\left[\left(1+\frac{1}{12p^2}\right)^s - 1\right] \leq \frac{sC}{10p^2},$$

which, by virtue of (3) and (4), proves the theorem.

If the partial derivatives of the function $f(x_1,\dots,x_s)$ satisfy the requirements of Theorem 1, then, by virtue of (5), the estimate

$$|C(m_1,\dots,m_s)| \leq \frac{C_1}{\left[(|m_1|+\delta_{m_1})\cdots(|m_s|+\delta_{m_s})\right]^2}$$

holds, where $C_1$ is some constant. Using more delicate number-theoretic estimates, one can, under substantially weaker requirements on the function $f(x_1,\dots,x_s)$, ensure the same order of accuracy of approximation of the integral by the corresponding sum as in Theorem 1:

**Theorem 2.** If there exist constants $C_1$ and $\varepsilon$ $(0<\varepsilon<1)$ such that for the Fourier coefficients of the function $f(x_1,\dots,x_s)$ the condition

$$|C(m_1,\dots,m_s)| \leq \frac{C_1}{\left[(|m_1|+\delta_{m_1})\cdots(|m_s|+\delta_{m_s})\right]^{1+\varepsilon}},$$

then for every prime $p > \frac{s}{\varepsilon}$, with $\xi_\nu(k) = \left\{\frac{k^\nu}{p}\right\}$ $(\nu = 1,2,\dots,s)$, for $N = p$ the estimate

$$\left|\int_0^1\cdots\int_0^1 f(x_1,\dots,x_s)\,dx_1\cdots dx_s - \frac{1}{N}\sum_{k=1}^{N} f[\xi_1(k),\dots,\xi_s(k)]\right| \leq \frac{(s-1)\sigma}{\sqrt{N}} + \frac{7sC_1}{\varepsilon N}.$$

In Theorems 1 and 2 the choice of points $M_k = M_k[\xi_1(k),\dots,\xi_s(k)]$, at which the values of the function $f(x_1,\dots,x_s)$ are computed, depends, obviously, on the value of $N$. There exist, however, such ways of choosing the points $M_k$ for which $N$ can be varied within comparatively wide limits without changing the positions of the chosen points.

**Theorem 3.** Let $p > s+1$ be prime and $\xi_\nu(k) = \left\{\frac{k^{\nu+1}}{p^2}\right\}$ $(\nu = 1,2,\dots,s)$. Then, under the assumption that the function $f(x_1,\dots,x_s)$ satisfies the conditions of Theorem 1, for every $N$ in the interval $p < N < p^2$ the estimate

$$\left| \int_0^1 \cdots \int_0^1 f(x_1, \ldots, x_s)\, dx_1 \cdots dx_s - \frac{1}{N} \sum_{k=1}^{N} f[\xi_1(k), \ldots, \xi_s(k)] \right| \le$$

$$\le \frac{(s-1)\sigma \ln N}{\theta N^{1-\frac{1}{2\theta}}} + \frac{sC}{10N^{\frac{1}{\theta}}}$$

holds (here $\theta$ is determined by the equality $N = p^{2\theta}$, and consequently $1/2 < \theta < 1$).

Analogous results can also be obtained from Theorem 2 by choosing in it $p > s+1$, $\xi_\nu(k) = \left\{ \frac{k^{\nu+1}}{p} \right\}$ and considering the interval $\sqrt{p} < N < p$.

The accuracy of the estimate in Theorem 3 decreases considerably for $N$ close to $p$. This drawback can be eliminated by defining the points $M_k[\xi_1(k), \ldots, \xi_s(k)]$, for example, in the following way: let $p_1 < p_2 < \cdots$ be consecutive primes; $p_1 > s$; for $t \ge 1$ $A(t) = p_1 + \cdots + p_t - t$; $A(0) = 0$; for $t = 1, 2, \ldots$, on the intervals $A(t-1) < k \le A(t)$, the quantities $\xi_\nu(k)$ are given by the equalities

$$\xi_\nu(k) = \left\{ \frac{k^\nu}{p_t} \right\} \qquad (\nu = 1, 2, \ldots, s). \tag{6}$$

With this choice of the points $M_k$, their position in the unit $s$-dimensional cube, obviously, does not depend on $N$. It is not difficult to show that, under the corresponding assumptions concerning $f(x_1, \ldots, x_s)$, the accuracy of the approximation of the integral by a sum of the form $\frac{1}{N} \sum_{k=1}^{N} f(M_k)$ has order not exceeding the quantity $\frac{1}{\sqrt[4]{N}}$ for any $N > p_1$.

**Remark.** In Theorems 1 and 2, and also in the equalities (6), one may, when choosing $\xi_\nu(k)$, without decreasing the accuracy of the estimates, use instead of the quantities $\left\{ \frac{k^\nu}{p^2} \right\}$, $\left\{ \frac{k^\nu}{p} \right\}$, $\left\{ \frac{k^\nu}{p_t} \right\}$ the quantities $\left\{ \frac{g^{\nu k}}{p^2} \right\}$, $\left\{ \frac{g^{\nu k}}{p} \right\}$, $\left\{ \frac{g_t^{\nu k}}{p_t} \right\}$, where $g$ and $g_t$ are primitive roots, respectively, modulo $p^2$ and $p_t$.

The question of approximate formulas convenient for computing multiple integrals was discussed in detail at the seminar on Monte Carlo methods at the Mathematical Institute of the Academy of Sciences of the USSR. The work of the seminar helped to reveal the most natural formulations of the problems whose solutions are given in the present paper. I take this opportunity to express my gratitude to the participants of the seminar and, especially, to N. N. Chentsov.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
19 VI 1957

* The proof of this and of the subsequent assertions is basically similar to the proof of Theorem 1.

*Note: Figure translations are in progress. See original paper for figures.*