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L. A. SAKHNOVICH

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Abstract

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MATHEMATICS

L. A. SAKHNOVICH

SPECTRAL ANALYSIS OF VOLTERRA OPERATORS AND INVERSE PROBLEMS

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In solving the inverse problem for the second-order differential equation

$$-\frac{d^2y}{dx^2} - q(x)y + \lambda y = 0 \qquad (0 \le x \le l) \tag{1}$$

V. A. Marchenko (^{1}) constructs a transformation operator which carries a solution of equation (1) into a solution of the simplest equation

$$\frac{d^2y}{dx^2} + \lambda y = 0 \qquad (0 \le x \le l). \tag{2}$$

If one passes to Green' s functions under the boundary conditions

$$y(0) = 0, \qquad y'(0) = 0,$$

then the differential operator

$$l(y) = \frac{d^2y}{dx^2} - q(x)y$$

will correspond to the integral operator

$$Kf = \int_0^x f(\xi)G(x,\xi)\,d\xi \qquad (0 \le x \le l), \tag{3}$$

where $G(x,x) = 0$, $\left.\frac{\partial}{\partial x}G(x,\xi)\right|_{\xi=x} = 1$.

From the results of V. A. Marchenko (^{1}) it follows easily that the operator K is linearly equivalent to the operator of repeated integration

$$J^2 f = \int_0^x f(\xi)(x-\xi)\,d\xi, \qquad f(\xi) \in L^2[0,l]. \tag{4}$$

In the present paper we find sufficient conditions under which a Volterra operator (not necessarily generated by a differential equation) is linearly equivalent to the operator of repeated integration (4). The results obtained are used in proving uniqueness theorems for systems of differential equations.

§ 1. Consider the Volterra operator

$$Kf = \int_0^x f(t)K(x,t)\,dt, \qquad f(x) \in L^2[0,l]. \tag{5}$$

In what follows we shall assume that the kernel $K(x,t)$ satisfies the following conditions:

1) the bounded derivatives exist

$$\frac{\partial^{j+k}}{\partial x^j \partial t^k} K(x,t) \quad (j,k = 0,1,2); \qquad \frac{d}{dx}\left[\frac{\partial^2}{\partial x^2} K(x,t)\Big|_{t=x}\right] \quad (x \geq t);$$

2) $K(x,x) \equiv 0$;

3)

$$\left|\frac{\partial}{\partial x} K(x,t)\Big|_{t=x}\right| > 0.$$

Without loss of generality one may assume that

$$\frac{\partial}{\partial x} K(x,t)\Big|_{t=x} \equiv 1.$$

Theorem 1. *If the kernel $K(x,t)$ of the operator K satisfies the conditions stated above, then the operator K is linearly equivalent to the operator J^2 of double integration:*

$$J^2 f = \int_0^x f(t)(x-t)\,dt, \qquad f(t) \in L^2[0,1].$$

Proof. Let $y(x,\lambda) = (I + \lambda^2 K)^{-1} f$; then

$$\frac{d^2 y}{dx^2} = \frac{d^2 f}{dx^2} - \lambda^2 y - \lambda^2 \int_0^x y(t,\lambda) \frac{\partial^2}{\partial x^2} K(x,t)\,dt. \tag{6}$$

Consider the operator

$$(I + H)f = f(x) + \int_0^x f(t)H(x,t)\,dt,$$

inverse to the operator

$$(I+K_2)f = f(x) + \int_0^x f(t)\frac{\partial^2}{\partial x^2}K(x,t)\,dt.$$

The equality holds

$$\frac{d^2y}{dx^2} = r(x) - \int_0^x \frac{d^2y}{dt^2}H(x,t)\,dt - \lambda^2 y, \tag{7}$$

where

$$r(x) = (I+H)\frac{d^2f}{dx^2}.$$

From the smoothness of $\frac{\partial^2}{\partial x^2}K(x,t)$ follows the smoothness of $H(x,t)$, and therefore we may write

$$\frac{d^2y}{dx^2} + \lambda^2 y = -q(x)\frac{dy}{dx} + y(x,\lambda)p(x) -$$
$$- \int_0^x y(t,\lambda)\frac{\partial^2}{\partial t^2}H(x,t)\,dt + s(x),$$

where

$$q(x) = H(x,x), \qquad \frac{\partial}{\partial t}H(x,t)\Big|_{t=x} = p(x),$$
$$s(x) = r(x) + \frac{dy}{dx}\Big|_{\lambda=0} H(x,0) - y(0)\,\frac{\partial}{\partial t}H(x,t)\Big|_{t=0}.$$

Choose $f(x)$ in equality (6) so that $s(x) \equiv 0$, $f(0)=1$, $\dfrac{df}{dx} = 0$; then

$$\frac{d^2y}{dx^2} + \lambda^2 y = -q(x)\frac{dy}{dx} + y(x,\lambda)p(x) - \int_0^x y(t,\lambda)\frac{\partial^2}{\partial t^2}H(x,t)\,dt.$$

Put

$$y = \exp\left[-\frac{1}{2}\int_0^x q(s)\,ds\right] y_1;$$

we obtain

$$\frac{d^2y_1}{dx^2} + \lambda^2 y_1 = y_1(x)u(x) - \int_0^x y_1(t,\lambda)\exp\left[\frac{1}{2}\int_t^x q(s)\,ds\right]\frac{\partial^2}{\partial t^2}H(x,t)\,dt, \tag{8}$$

where $u(x) = p(x) + \frac{1}{2}(q(x)^2 + q'(x))$.

In the paper (¹) an operator is constructed which transforms the solution of equation (8), when $\frac{\partial^2}{\partial t^2} H(x,t) \equiv 0$, into the solution of the equation

$$\frac{d^2 y_1}{dx^2} + \lambda^2 y_1 = 0. \tag{9}$$

Repeating almost verbatim the arguments of V. A. Marchenko, one can, in the general case, construct an operator transforming the solution of equation (8) into the solution of equation (9), i.e. the equality

$$y_1(x,\lambda) = \cos \lambda x + \int_0^x \cos \lambda t \cdot N(x,t)\, dt,$$

where $|N(x,t)| \leq N$ $(0 \leq t \leq x \leq l)$, will hold.

Then

$$y(x,\lambda) = V \cos \lambda x = \left[\cos \lambda x + \int_0^x \cos \lambda t \cdot N(x,t)\, dt\right] \exp\left[-\frac{1}{2}\int_0^x q(s)\, ds\right], \tag{10}$$

i.e.

$$(I + \lambda^2 K)^{-1} f = V(I + \lambda^2 J^2)^{-1} \cdot 1. \tag{11}$$

Expanding both sides of equality (11) in a series in λ and equating the corresponding coefficients, it is not difficult to verify the validity of the relation

$$K = V J^2 V^{-1}.$$

Theorem 1 is proved.

Remark. Theorem 1 remains valid if the operator K (1) is given in the space $L_m^2[0,l]$* and $K(x,t)$ is a matrix-function. In this case, instead of condition 3), one should require

$$\frac{\partial}{\partial x} K(x,t)\Big|_{t=x} \equiv I,$$

and the operator J^2 should also be regarded as given in $L_m^2[0,l]$.

Let us note that operator (5) has a system of nested invariant subspaces H_α, where H_α consists of all vector-functions vanishing on the interval $[0,\alpha]$ $(\alpha > 0)$. Naturally the question arises whether operator (5) has other invariant subspaces, distinct from H_α.

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* By the space $L_m^2[0,l]$ we shall mean the space of vector-functions

$$f(x) = [f_1(x), f_2(x), \dots, f_m(x)]$$

with scalar product

$$(f, g) = \sum_{i=1}^{m} \int_0^l f_i(x)\overline{g_i(x)}\,dx.$$

From Titchmarsh’ s theorem (2), which concerns integral equations of the form

$$\int_0^x K(x-y)f(y)\,dy = 0, \qquad f(y) \in L^2[0,l],$$

it follows easily that J^2 has no invariant subspaces other than H_α. From Theorem 1 it then follows that the operator $K|1|$ also has no invariant subspaces other than H_α.

Corollary. The function $f(x)$ is a generating function for the operator $K|1|$ if and only if, for every α ($\alpha > 0$), in the segment $[0, \alpha]$ there exists a set of points of positive measure on which $f(x) \neq 0$.

§ 2. Consider the system of differential equations

$$\frac{dW}{dx} = i\lambda W\beta^2(x)J, \qquad W(0,\lambda) = I \quad (0 \le x \le l), \tag{12}$$

where $\beta^2(x)$ is a Hermitian nonnegative matrix of finite or infinite order r; J is a diagonal matrix whose diagonal entries are either $+1$ or -1.

We shall assume that $\operatorname{rang}\beta^2(x) \equiv 1$; then the matrices $\beta(x)$ and $\beta^2(x)J$ are represented in the form

$$\beta(x) = \|\varphi_i(x)\overline{\varphi_j(x)}\|_{i,j=1}^{r},$$

$$\beta^2(x)J = \sum_{k=1}^{r} |\varphi_k(x)|^2 \cdot \|\varphi_i(x)\overline{\varphi_j(x)}\varepsilon_j\|_{i,j=1}^{r}, \tag{13}$$

where $\varepsilon_i = \pm 1$.

Let $\beta^2(x)J$ have the following properties:

1)

$$\sum_{j=1}^{r} |\varphi_j(x)|^2 \leq M \qquad (0 \leq x \leq l);$$

2) the kernel

$$K(x,t) = i\sum_{j=1}^{r} \varphi_j(x)\overline{\varphi_j(t)}\varepsilon_j$$

satisfies the conditions of Theorem 1.

From a work of M. S. Livshits (3) it follows that system (12) is associated with the operator

$$Kf = \int_0^x f(t)K(x,t)\,dt$$

and the matrix $W(l, 1/\lambda)$ coincides with the characteristic matrix-function of the operator K, i.e. the matrix $W(l, 1/\lambda)$ characterizes the operator K up to unitary equivalence. Using this fact and Theorem 1, we have proved the following proposition:

Theorem 2. If $\beta^2(x)J$ is represented in the form (13) and satisfies conditions 1), 2), then the matrix $W(l, \lambda)$ determines the system of differential equations uniquely.

In the special case ($r = 2$), this yields the known results (1) on the uniqueness of the solution of the inverse problem for an ordinary differential equation of second order on a finite interval.

Odessa Pedagogical Institute
named after K. D. Ushinsky

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Note: Figure translations are in progress. See original paper for figures.

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