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D. D. IVLEV

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Abstract

Full Text

THEORY OF ELASTICITY

D. D. IVLEV

APPROXIMATE SOLUTION OF ELASTO-PLASTIC PROBLEMS OF THE THEORY OF IDEAL PLASTICITY

(Presented by Academician A. I. Nekrasov, 12 X 1956)

We shall seek the solution of the elastoplastic problem in the form of series in powers of a certain parameter δ:

$$\sigma_{ij} = \sum_{n=0} \delta^n \sigma_{ij}^{(n)}, \qquad \sigma_{ii} \equiv \sigma_\rho, \qquad \sigma_{jj} \equiv \sigma_\theta, \qquad \sigma_{ij} \equiv \sigma_{ji} \equiv \tau_{\rho\theta}. \tag{1}$$

We shall confine ourselves to consideration, in the theory of plane strain, of the plasticity conditions of Mises and Saint-Venant ([1]), which are essentially coincident:

$$\frac{1}{4}(\sigma_\rho - \sigma_\theta)^2 + \tau_{\rho\theta}^2 = 1; \tag{2}$$

in the theory of plane stress—the Saint-Venant plasticity condition ([1]):

$$\frac{1}{4}(\sigma_\rho - \sigma_\theta)^2 + \tau_{\rho\theta}^2 = \left[1 - \frac{1}{2}(\sigma_\rho + \sigma_\theta)\right]^2 \quad \text{for } \sigma_\theta > \sigma_\rho > 0. \tag{3}$$

In (1), (2), (3) all quantities are dimensionless. Substituting (1) into (2), with $\tau_{\rho\theta}^0 = 0$, we obtain:

$$\sigma_\rho' - \sigma_\theta' = 0, \qquad (\sigma_\rho'' - \sigma_\theta'')\mu + \tau_{\rho\theta}'^2 = 0, \qquad \frac{1}{2}(\sigma_\rho''' - \sigma_\theta''')\mu + \tau_{\rho\theta}' \tau_{\rho\theta}'' = 0, \tag{4}$$

where $\mu = \operatorname{sign}(\sigma_\rho^0 - \sigma_\theta^0)$.

Substituting (1) into (3) with $\tau_{\rho\theta}^0 = 0$, we obtain:

$$\sigma_\theta' = 0, \qquad (1 - \sigma_\rho^0)\sigma_\theta'' + \tau_{\rho\theta}'^2 = 0, \qquad (1 - \sigma_\rho^0)\sigma_\theta''' - \sigma_\rho' \sigma_\theta'' + 2\tau_{\rho\theta}' \tau_{\rho\theta}'' = 0. \tag{5}$$

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If on the boundary L of the body under consideration

$$\sigma_{\text{n}} = P_{\text{n}}, \qquad \tau_{\text{n}} = P_{\tau}, \tag{6}$$

then, assuming the equation of the contour L to be given in the form

$$\rho = \sum_{n=0} \delta^n \rho_n(\theta),$$

we obtain the linearized boundary conditions (6) in the form

$$\sigma'_\rho + \frac{d\sigma^0_\rho}{d\rho}\rho_1 = \frac{dP_{\text{n}}}{d\rho}\rho_1; \qquad \tau'_{\rho\theta} - (\sigma^0_\theta - \sigma^0_\rho)\dot{R}_1 = \frac{dP_\tau}{d\rho}\rho_1;$$

$$\sigma''_\rho + \frac{d\sigma'_\rho}{d\rho}\rho_1 + \frac{d^2\sigma^0_\rho}{d\rho^2}\frac{\rho_1^2}{2!} + \frac{d\sigma^0_\rho}{d\rho}\rho_2 + (\sigma^0_\theta - \sigma^0_\rho)\dot{R}_1^2 - 2\tau'_{\rho\theta}\dot{R}_1 =$$

$$= \frac{d^2P_{\text{n}}}{d\rho^2}\frac{\rho_1^2}{2!} + \frac{dP_{\text{n}}}{d\rho}\rho_2;$$

$$\begin{gathered}\tau''_{\rho\theta} - (\sigma^0_\theta - \sigma^0_\rho)(\dot{R}_2 - R_1\dot{R}_1) - (\sigma'_\theta - \sigma'_\rho)\dot{R}_1 + \frac{d}{d\rho}\left[\tau'_{\rho\theta} - (\sigma^0_\theta - \sigma^0_\rho)\dot{R}_1\right]\rho_1 = \\ = \frac{d^2P_\tau}{d\rho^2}\frac{\rho_1^2}{2!} + \frac{dP_\tau}{d\rho}\rho_2;\end{gathered} \tag{7}$$

$$\begin{gathered}\sigma'''_\rho + \frac{d\sigma''_\rho}{d\rho}\rho_1 + \frac{d^2\sigma'_\rho}{d\rho^2}\frac{\rho_1^2}{2!} + \frac{d^3\sigma^0_\rho}{d\rho^3}\frac{\rho_1^3}{3!} + 2(\sigma^0_\theta - \sigma^0_\rho)(\dot{R}_1\dot{R}_2 - R_1\dot{R}_1^2) + \\ + \frac{d(\sigma_\theta - \sigma_\rho)}{d\rho}\dot{R}_1^2\rho_1 - (\sigma'_\theta - \sigma'_\rho)\dot{R}_1^2 - 2\tau'_{\rho\theta}(\dot{R}_2 - R_1\dot{R}_1) - \\ -2\frac{d\tau_{\rho\theta}}{d\rho}\dot{R}_1\rho_1 - 2\tau''_{\rho\theta}\dot{R}_1 + \frac{d\sigma'_\rho}{d\rho}\rho_2 + \frac{d^2\sigma^0_\rho}{d\rho^2}\rho_1\rho_2 + \frac{d\sigma^0_\rho}{d\rho}\rho_3 = \\ = \frac{d^3P_{\text{n}}}{d\rho^3}\frac{\rho_1^3}{3!} + \frac{d^2P_{\text{n}}}{d\rho^2}\rho_1\rho_2 + \frac{dP_{\text{n}}}{d\rho}\rho_3;\end{gathered}$$

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$$\tau'''_{\rho\theta} - 2\tau'_{\rho\theta}\dot{R}_1^2 - (\sigma_\theta^0 - \sigma_\rho^0)(\dot{R}_3 - R_1\dot{R}_2 + R_1^2\dot{R}_1 - \dot{R}_1R_2 - \dot{R}_1^3) -$$
$$-(\sigma'_\theta - \sigma'_\rho)(\dot{R}_2 - R_1\dot{R}_1) - (\sigma''_\theta - \sigma''_\rho)\dot{R}_1 + \frac{d}{d\rho}\left[\tau''_{\rho\theta} - (\sigma_\theta^0 - \sigma_\rho^0)(\dot{R}_2 - R_1\dot{R}_1) -\right.$$
$$\left.-(\sigma'_\theta - \sigma'_\rho)\dot{R}_1\right]\rho_1 + \frac{d^2}{d\rho^2}\left[\tau'_{\rho\theta} - (\sigma_\theta^0 - \sigma_\rho^0)\dot{R}_1\right]\frac{\rho_1^2}{2!} + \frac{d}{d\rho}\left[\tau'_{\rho\theta} - (\sigma_\theta^0 - \sigma_\rho^0)\dot{R}_1\right]\rho_2 =$$
$$= \frac{d^3P_\tau}{d\rho^3}\frac{\rho_1^3}{3!} + \frac{d^2P_\tau}{d\rho^2}\rho_1\rho_2 + \frac{dP_\tau}{d\rho}\rho_3,$$

where $R_i = \rho_i/\rho_0$.

Since on the boundary of the plastic region L_s the solutions for the plastic and elastic regions are matched continuously (1):

$$[\sigma_\rho] = [\sigma_\theta] = [\tau_{\rho\theta}] = 0 \quad \text{on } L_s, \tag{8}$$

then, representing the equation of the contour L_s in the form

$$\rho_s = \sum_{n=0}\delta^n\rho_{sn}(\theta), \tag{9}$$

we obtain the linearized matching conditions (8) in the form

$$\left[\sigma'_{ij} + \frac{d\sigma_{ij}^0}{d\rho}\rho_{s1}\right] = 0, \qquad \left[\sigma''_{ij} + \frac{d\sigma'_{ij}}{d\rho}\rho_{s1} + \frac{d^2\sigma_{ij}^0}{d\rho^2}\frac{\rho_{s1}^2}{2!} + \frac{d\sigma_{ij}^0}{d\rho}\rho_{s2}\right] = 0;$$
$$\left[\sigma'''_{ij} + \frac{d\sigma''_{ij}}{d\rho}\rho_{s1} + \frac{d^2\sigma'_{ij}}{d\rho^2}\frac{\rho_{s1}^2}{2!} + \frac{d^3\sigma_{ij}^0}{d\rho^3}\frac{\rho_{s1}^3}{3!} + \frac{d\sigma_{ij}^0}{d\rho}\rho_{s3} + \frac{d\sigma'_{ij}}{d\rho}\rho_{s2} + \right. \tag{10}$$
$$\left. + \frac{d^2\sigma_{ij}^0}{d\rho^2}\rho_{s1}\rho_{s2}\right] = 0 \text{ etc.}$$

We note that from the equilibrium equations and conditions (8) it follows that

$$\left[\frac{d\sigma_\rho^0}{d\rho}\right] = 0 \quad \text{on } L_s.$$

Consequently, the matching conditions (10) for σ_ρ and $\tau_{\rho\theta}$ play the role of boundary conditions for determining the stresses in the elastic region, while the matching condition for σ_θ serves to determine ρ_{sn}. The greatest difficulty and interest in such problems is the determination of the equation of the boundary of the plastic zone L_s. We shall present several approximate solutions.

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1. **Biaxial tension of a thick plate with a circular hole of radius a by forces P_1 and P_2.** Denoting

$$\delta = \frac{|P_1 - P_2|}{2k}, \tag{11}$$

where $k = \frac{1}{4}\sigma_s$ according to Saint-Venant and $k = \frac{1}{\sqrt{3}}\sigma_s$ according to Mises, one may obtain:

$$\rho_s = 1 + \delta\cos 2\theta - \frac{3}{4}\delta^2(1 - \cos 4\theta) + \frac{5}{8}\delta^3(-\cos 2\theta + \cos 6\theta) + \\ + \frac{7}{64}\delta^4(-1 - 4\cos 4\theta + 5\cos 8\theta) + \cdots \tag{12}$$

Expansion (12) coincides exactly with the expansion of the equation of an ellipse with semiaxes $(1+\delta)$, $(1-\delta)$, which, as shown in ([2]), is the exact solution of this problem.

2. **Biaxial tension of a thin plate with a circular hole of radius a by forces P_1 and P_2.** Introducing δ (11), one may obtain:

$$\rho_s = 1 + 4\delta^*\cos 2\theta - 8\delta^{*2}(1 - 2\cos 4\theta) - 80\delta^{*3}(\cos 2\theta - \cos 6\theta) + \\ + 32\delta^{*4}(1 - 16\cos 4\theta + 14\cos 8\theta) + \cdots, \tag{13}$$

where $\delta^* = \delta/\alpha$; $\alpha = a/r_s^0$; r_s^0 is the size of the radius of the plastic zone for $\delta = 0$. The first approximation (13) was obtained in ([3]).

3. **Biaxial tension of a thin plate with an elliptical hole by forces $P_1 d_2$ and $P_2 d_2$, directed at an angle θ_0 to the principal central axes of the ellipse.** Introducing δ (11), one may obtain:

$$\rho_s = 1 + \delta^*(4d_2\cos 2(\theta - \theta_0) + 3\alpha d_1\cos 2\theta) + \delta^{*2}\{d_1^2(\alpha^2/4 - 8\alpha^4) - \\ -(18d_1d_2\alpha\cos 2\theta_0 + 8d_2^2) + [-d_1^2(15/4\,\alpha^2 - 8\alpha^3 - 3/4\,\alpha^4) + \\ +(18d_1d_2\alpha\cos 2\theta_0 + 16d_2^2\cos 4\theta_0)]\cos 4\theta + \\ +[18d_1d_2\alpha\sin 2\theta_0 + 16d_2^2\sin 4\theta_0]\sin 4\theta\} + \cdots,$$

where the equation of the ellipse of the hole is represented in the form

$$\rho = \alpha + \delta\alpha d_1\cos 2\theta - \delta^2\frac{3\alpha d_1^2}{4}(1 - \cos 4\theta) + \cdots.$$

For $d_1 = 0,\ \theta_0 = 0,\ d_2 = 1$ there is the case of biaxial tension of a thin plate with a circular hole; for $d_2 = 0,\ d_1 = 1$, the case of uniform tension of a thin plate with an elliptical hole.

4. **An eccentric tube under the action of internal pressure p_0.** Referring all linear quantities to the outer radius of the tube b, denote $\delta = c/b$, where c is the eccentricity of the tube. One may obtain:

$$\rho_s = \beta_0 - \delta\frac{2\beta_0^4}{1-\beta_0^4}\cos\theta + \delta^2\left\{\frac{\beta_0^3(2-\beta_0^4-\beta_0^6)}{(1-\beta_0^2)(1-\beta_0^4)^2} + \frac{2\beta_0^7}{(1-\beta_0^4)^2} + \right.$$
$$+\frac{1}{N}\left[-\frac{(1-\beta_0^2)(1-3\beta_0^4)}{\beta_0} + \frac{(1-\beta_0^2)^2(5+3\beta_0^4)\beta_0}{(1-\beta_0^4)} - \right.$$
$$\left.\left. -\frac{\beta_0^3}{(1-\beta_0^4)^2}[(1+\beta_0^2)^4 + 4\beta_0^4(2-\beta_0^2)^2 - 4(1+4\beta_0^4)] + \frac{2\beta_0^3(1-\beta_0^2)^2}{(1+\beta_0^2)^2}\right]\cos 2\theta\right\},$$

where $\beta_0 = r_s^0/b,\ N = (\beta_0 - 1/\beta_0)^4$.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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