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Abstract

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ON APPROXIMATE METHODS FOR SOLVING DIFFERENTIAL EQUATIONS IN BANACH SPACE

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Consider the equation

$$\frac{dx}{dt} + A(t)x = f(t,x) \qquad (0 \le t \le T), \tag{1}$$

where $x(t)$ is the unknown function with values in a Banach space E; $A(t)$ and $f(t,x)$, for each $t \in [0,T]$, are operators acting in E. We shall assume that $A(t)$ is a linear unbounded closed operator with domain $D(A)$, independent of t, and that $f(t,x)$ is a bounded nonlinear operator.

In papers ([1,2]), under certain assumptions, it was established that there exists a solution $x(t)$ of equation (1), defined on some segment $[0,h]$, and satisfying the initial condition

$$x(0) = x^0 \qquad (x^0 \in D(A)). \tag{2}$$

Let $A_n(t)$ be bounded operators approximating uniformly in t the operator $A(t)$ on its domain of definition:

$$\lim_{n\to\infty} \sup_{0\le t\le T} \|[A_n(t) - A(t)]x\| = 0 \qquad (x \in D(A)). \tag{3}$$

Let the bounded operators $f_n(t,x)$ converge uniformly in t to the operator $f(t,x)$ for each x from some ball S with center at the point x^0. Finally, let x_n^0 converge to x^0. In the present paper we consider the question of under what conditions the solutions $x_n(t)$ of the equations

$$\frac{dx}{dt} + A_n(t)x = f_n(t,x), \tag{4}$$

satisfying the initial conditions

$$x_n(0) = x_n^0, \tag{5}$$

converge to the solution $x(t)$ of problem (1)–(2).

1. The homogeneous equation with a constant operator

$$\frac{dx}{dt} + Ax = 0 \tag{6}$$

has the solution $x(t) = e^{-tA}x^0$, satisfying the initial condition (2), if the operator $-A$ is the infinitesimal generator of a strongly continuous semigroup ([3]).

In the case of a constant operator A, it is natural also to choose the approximating operators A_n to be constant, and to write the equation for the approximate solutions in the form

$$\frac{dx}{dt} + A_{nx} = 0. \tag{7}$$

The solutions of these equations under the initial conditions (5) are given by the formula:

$$x_n(t) = e^{-tA_n}x_n^0.$$

Theorem 1. In order that the sequence of operators e^{-tA_n} converge strongly to the operator e^{-tA}, uniformly in $t \in [0, T]$, it is necessary and sufficient that the inequality

$$\sup_{0 \leq t \leq T} \|e^{-tA_n}\| \leq M,$$

hold, where M does not depend on n.

The assertion of Theorem 1 is proved in the book (4) under the additional assumption that the operators A_n commute. This assumption is not satisfied for the operators A_n encountered in most approximate methods.

It follows from Theorem 1 that $x_n(t)$ converge to $x(t)$ uniformly in $t \in [0, T]$. If $\|A_n x_n^0 - Ax_0\| \to 0$, then, moreover, dx_n/dt also converge to dx/dt uniformly in $t \in [0, T]$.

2. Let us now consider the homogeneous equation

$$\frac{dx}{dt} + A(t)x = 0. \tag{8}$$

We shall assume that for every $t \in [0, T]$

$$\|[\lambda I + A(t)]^{-1}\| \leq \frac{1}{1+\lambda} \quad (\lambda > -1) \tag{9}$$

and that the operator $C(t) = A(t)\, dA^{-1}(t)/dt$ is bounded and strongly continuous with respect to t.

It follows from the results of (5) that under these conditions there exists an operator $U(t,s)$ $(t \geq s)$, strongly continuous jointly in t and s, and such that the function $x(t) = U(t,s)x^0$, for $x^0 \in D(A)$, satisfies, for $t \geq s$, equation (8) and the initial condition $x(s) = x^0$. Thus, under the initial condition (2), the solution of equation (8) has the form $x(t) = U(t,s)x^0$.

Consider the equation for approximate solutions

$$\frac{dx}{dt} + A_n(t)x = 0. \tag{10}$$

If the bounded operator $A_n(t)$ is strongly continuous in t, then equation (10) has a solution for any initial condition x_n^0. This solution, as in the case of the operator $A(t)$, can be written in the form $x_n(t) = U_n(t,0)x_n^0$, where the operator $U_n(t,s)$ has properties analogous to those of the operator $U(t,s)$.

Theorem 2. Suppose that all the operators $A_n(t)$ satisfy condition (9). Then the operators $U_n(t,s)$ converge strongly to the operator $U(t,s)$, uniformly in t and s for $0 \leq s \leq t \leq T$.

The proof is based on the formula

$$[U(t,s) - U_n(t,s)]x = \int_s^t U_n(t,\tau)[A(\tau) - A_n(\tau)]U(\tau,0)x\, d\tau, \tag{11}$$

valid for every $x \in D(A)$.

Let us note that from Theorem 2 there follow assertions analogous to those given at the end of the preceding section.

3. In (1), the continuous solutions of the integral equation

$$x(t) = U(t,0)x^0 + \int_0^t U(t,s)f(s,x(s))\, ds \tag{12}$$

were called **generalized solutions of equation** (1) under the initial condition (2). This name is justified by the fact that generalized solutions are ordinary solutions if certain smoothness assumptions on $f(t,x)$ are fulfilled. In the case of a bounded operator $A(t)$, equation (12) and problem (1)–(2) are equivalent.

Theorem 3. *Let the operators $A(t)$ and $A_n(t)$ satisfy the conditions of items 1 or 2, according as whether or not they depend on t. Let the operators $f_n(t,x)$*

be continuous in $t \in [0, T]$ and, in the ball S, satisfy a Lipschitz condition in x with a constant independent of t and n. Then, beginning with some n, problem (4)–(5) has solutions defined on some segment $[0, h]$. These solutions converge uniformly in $t \in [0, h]$ to the generalized solution of equation (1) under the initial condition (2).

Theorem 3 may be regarded as a justification of various approximate methods for solving equation (1).

4. In what follows equation (1) is considered in a separable Hilbert space H.

The Bubnov–Galerkin method for the approximate solution of problem (1)–(2) consists in replacing equation (1) by the equations

$$\frac{dx}{dt} + P_n A(t)x = P_n f(t, x), \tag{13}$$

where P_n is the operator of orthogonal projection onto the linear span R_n of the first n elements of the basis $\{e_n\}$, while the initial condition (2) is replaced by the initial condition (5), with x_n^0 chosen from R_n so that $\|x_n^0 - x^0\| \to 0$. Suppose that the basis $\{e_n\}$ is composed of eigenvectors of a certain self-adjoint operator C, for which $D(C) = D(A)$.

Theorem 4. *Let the operator $A(t)$ satisfy the conditions of item 2, and let the operator $f(t, x)$ be continuous in t on $[0, T]$ and, in the ball S, satisfy a Lipschitz condition in x with a constant independent of t. Then, beginning with some n, problem (13)–(5) has solutions defined on some segment $[0, h]$. These solutions converge uniformly in $t \in [0, h]$ to the generalized solution of equation (1) under the initial condition (2).*

In proving that the operators $P_n A(t)$ satisfy condition (9) on R_n, we used the following lemma.

Lemma. *For (9) to hold it is necessary and sufficient that*

$$\operatorname{Re}(Ax, x) \geq (x, x) \quad (x \in D(A)); \qquad \operatorname{Re}(A^*x, x) \geq (x, x) \quad (x \in D(A^*)).$$

The sufficiency of these conditions was proved by V. E. Lyantse [6]; the proof of their necessity was communicated to us by S. G. Krein.

5. Consider the equation:

$$\frac{dx}{dt} + Ax = f(t, x, Bx), \tag{14}$$

where A is a self-adjoint, positive-definite operator; B is a closed operator with $D(B) \supset D(A^{1/2})$. Let a continuous function $y(t)$ satisfy the equation

$$y(t) = f\left(t, e^{-tA}x^0 + \int_0^t e^{-(t-s)A}y(s)\,ds,\ Be^{-tA}x^0 + B\int_0^t e^{-(t-s)A}y(s)\,ds\right). \tag{15}$$

Then the function

$$x(t) = e^{-tA}x^0 + \int_0^t e^{-(t-s)A}y(s)\,ds$$

is naturally called a **generalized solution of equation** (14)

under the initial condition (2), since it is its ordinary solution, provided certain smoothness assumptions on $f(t, x, y)$ are satisfied ([2]).

Application of the Bubnov–Galerkin method to equation (14) leads to the solution of the equations

$$\frac{dx}{dt} + P_{nAx} = P_{nf}(t, x, Bx). \tag{16}$$

Theorem 5. Suppose that the operator $f(t, x, y)$ is continuous in $t \in [0, T]$ and, in the ball S containing x_0 and Bx_0 in its interior, satisfies a Lipschitz condition in the aggregate of the variables x and y, with a constant independent of t; suppose that $\|A^{1/2}[x_n^0 - x^0]\| \to 0$. Then, starting from some n, problem (16)–(5) has solutions $x_n(t)$ defined on some segment $[0, h]$. These solutions converge uniformly in t on $[0, h]$ to the generalized solution of equation (14) under the initial condition (2).

If the basis $\{e_n\}$ consists of eigenvectors of the operator A, then Theorem 5 admits a substantial strengthening. In this case it is sufficient to require of the operator B that $D(B)$ contain $D(A^{1-\alpha})$, where α is an arbitrarily small positive number. The assertion of Theorem 5 then holds if $\|A^{1-\alpha}[x_n^0 - x^0]\| \to 0$.

We note that another approach to the study of the convergence of the Bubnov–Galerkin method for equations of the form (1) may be found in the works ([7–9]).

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Note: Figure translations are in progress. See original paper for figures.