MATHEMATICS

S. F. PASHKOVSKII

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Abstract

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S. F. PASHKOVSKII

ON THE LOCATION OF (e)-POINTS OF POLYNOMIALS OF BEST APPROXIMATION

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1. In the present note W_n denotes the class of all algebraic polynomials of degree not exceeding n. For any function f, continuous on the segment $[-1, 1]$ (closed), the norm is defined by

$$\|f\|_{[-1,1]} = \max_{|t|\le 1} |f(t)|.$$

The error of best approximation of the function f by polynomials of the class W_n is the number

$$E_n(f) = \min_{p\in W_n} \|f - p\|_{[-1,1]}.$$

The polynomial p_n of best approximation to the function f in the class W_n, by definition, satisfies the equality

$$\|f - p_n\|_{[-1,1]} = E_n(f).$$

We call an (e)-point of the polynomial p_n any point $u \in [-1, 1]$ at which

$$|f(u) - p_n(u)| = E_n(f).$$

It is known that the polynomial p_n has no fewer than $n + 2$ (e)-points $u_0, u_1, \dots, u_{n+1}$ (where $-1 \le u_0 < u_1 < \dots < u_{n+1} \le 1$), at which the values of the function $f - p_n$ are alternately positive and negative.

2. Polynomials of best approximation are conveniently computed, especially on electronic digital machines, by the mesh variant of the method of leveling maxima due to Remez ([1]), Part II. In this variant the amount of computation depends on the number of mesh points, and therefore depends on preliminary knowledge of the location of the (e)-points in the interval of approximation. Indeed, if nonintersecting segments $I_0, I_1, \dots, I_{n+1}$ are known to which the (e)-points $u_0, u_1, \dots, u_{n+1}$, respectively, belong, then instead of considering the given function f on the segment $[-1, 1]$, we may restrict ourselves to considering it on the sum of the segments $I_0, I_1, \dots, I_{n+1}$, since the polynomials of best approximation to f in the class W_n, constructed for the segment $[-1, 1]$ and for the set

$$I_0 \cup I_1 \cup \dots \cup I_{n+1},$$

coincide.

Until now there have existed no theorems from which one could obtain the above-mentioned nonintersecting segments $I_0, I_1, \dots, I_{n+1}$.

3. In this note I present estimates of the (e)-points which, for sufficiently small values of the quotient

$$E_{n+1}(f)/E_n(f),$$

give segments I_k already possessing the indicated property.

Below, $T_{n,k,h}$ (where n and $k \leq n+1$ are natural numbers, $0 < h \leq 1$) denotes a polynomial of degree $n+1$, all of whose roots are real, simple, and lie in the interval $(-1, 1)$ (open), and for which

$$T_{n,k,h}(v_i) = \begin{cases} (-1)^{n+1-i}h, & \text{for } i = 0, 1, \dots, k-1, \\ (-1)^{n+1-i}, & \text{for } i = k, k+1, \dots, n+1, \end{cases}$$

where $v_0 = -1$, $v_{n+1} = 1$, and the numbers

$$v_1 < v_2 < \dots < v_n$$

are the roots of the derivative of the polynomial $T_{n,k,h}$.

The largest among those roots of the equation

$$|T_{n,k,h}(t)| = h$$

which are less than v_k is denoted by $t_{n,k,h}$. In addition, we shall put

$$t_{n,0,h} = -1$$

for all n and h.

Theorem 1. For all continuous functions f such that

$$\frac{E_{n+1}(f)}{E_n(f)} \leqslant g < 1, \tag{1}$$

the inequalities

$$t_{n,k,h} \leqslant u_k \leqslant -t_{n,n+1-k,h} \qquad (k = 0, 1, \dots, n+1),$$

are satisfied, where $h = (1-g)/(1+g)$.

This theorem follows easily from the following two lemmas, of which the first belongs to approximation theory, and the second to the theory of algebraic polynomials.

Lemma 1. For all continuous functions f possessing property (1), the relations

$$u_k \in [z_k, z_{k+1}] \cap E_t\{|p_n(t) - p_{n+1}(t)| \geqslant h\|p_n - p_{n+1}\|_{[-1,1]}\}$$

$$(k = 0, 1, \dots, n+1),$$

hold, where p_n and p_{n+1} are the polynomials of best approximation to the function f, respectively in the classes W_n and W_{n+1}, and z_k is the k-th, in magnitude, root of the polynomial $p_n - p_{n+1}$. In Lemma 1 the following easily proved fact is used: under assumption (1), the difference $p_n - p_{n+1}$, which is a polynomial of degree $n+1$, has only real, simple roots belonging to the interval $(-1, 1)$.

Lemma 2. Let P be any polynomial of degree $n+1$ with real roots $z_1^P, z_2^P, \dots, z_{n+1}^P$, where

$$-1 \equiv z_0^P < z_1^P < z_2^P < \dots < z_{n+1}^P < z_{n+2}^P \equiv 1.$$

Let

$$\min_{0 \leqslant k \leqslant n+1} \|P\|_{[z_k^P, z_{k+1}^P]} \geqslant h\|P\|_{[-1,1]}, \tag{2}$$

$$[t_k^P, \bar{t}_k^P] = [z_k^P, z_{k+1}^P] \cap E_t\{|P(t)| \geqslant h\|P\|_{[-1,1]}\} \qquad (k = 0, 1, \dots, n+1).$$

Then

$$\inf_P t_k^P = t_{n,k,h}, \qquad \sup_P \bar{t}_k^P = -t_{n,n+1-k,h} \qquad (k = 0, 1, \dots, n+1),$$

where the lower and upper bounds are extended over all polynomials P possessing property (2).

4. To apply Theorem 1 in practice, it was necessary to compute the values $t_{n,k,h}$ for various n, k, and h. Such computations have so far been carried out for $n = 2, 3, 4, 5, 6$, and 7, at the Computing Center of Moscow State University on the electronic digital computer "Strela." The computations showed, among other things, that the segments

$$I_k = [t_{n,k,h}, -t_{n,n+1-k,h}]$$

do not intersect when

$$\frac{E_{n+1}(f)}{E_n(f)} \leqslant \begin{cases} 0.374 & \text{for } n = 2, \\ 0.358 & \text{for } n = 3, \\ 0.334 & \text{for } n = 4, \\ 0.324 & \text{for } n = 5, \\ 0.313 & \text{for } n = 6, \\ 0.303 & \text{for } n = 7. \end{cases}$$

Consequently, the theorem can be applied to a fairly broad class of continuous functions f.

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References

1. E. Ya. **Remez**, *General Computational Methods of Chebyshev Approximation*, Kiev, 1957.

Note: Figure translations are in progress. See original paper for figures.