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Abstract

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ONE CLASS OF INFINITE SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

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In the present note we study infinite systems of linear equations with unknowns x_k:

$$x_n = \sum_{k=-\infty}^{\infty} a_{n-k} x_k + b_n \quad (n = \dots, -2, -1, 0, 1, \dots), \tag{1}$$

$$x_n = \sum_{k=0}^{\infty} a_{n-k} x_k + b_n \quad (n = 0, 1, 2, \dots). \tag{2}$$

We shall base the investigation on the analogy between these systems and the integral equations

$$y(x) = \int_{-\infty}^{\infty} K(x-s) y(s)\, ds + f(x); \tag{3}$$

$$y(x) = \int_{0}^{\infty} K(x-s) y(s)\, ds + f(x), \quad x > 0. \tag{4}$$

§ 1. Consider sequences $\{a_n\}$ $(n = \dots, -1, 0, 1, \dots)$, for which

$$R = \left[\lim \sqrt[n]{|a_n|}\right]^{-1} \neq 0, \quad r = \lim \sqrt[n]{|a_{-n}|} \neq \infty.$$

a) In the case $R > r$, let us associate with the sequence $\{a_n\}$ the Laurent series

$$\sum_{k=-\infty}^{\infty} a_k z^k,$$

which represents a function regular in the annulus $r < |z| < R$. We shall agree to call the sequence $\{a_n\}$ and the function $A(z)$ **Laurent transforms**, the former the **inverse transform**, the latter the **direct transform**. Briefly, we denote this correspondence by the symbol $a_n \to A(z)$.

Clearly,

$$a_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{A(z)\,dz}{z^{n+1}}, \quad r < \rho < R \quad (n = \ldots, -1, 0, 1, \ldots).$$

b) In the case $R < r$, we shall consider the generalized transforms

$$A^+(z) = \sum_{k=0}^{\infty} a_k z^k, \quad A^-(z) = \sum_{k=-\infty}^{-1} a_k z^k.$$

Then a_n is expressed through $A^+(z)$

and $A^-(z)$ by the formula

$$a_n = \frac{1}{2\pi i} \int_{|z|=\rho_1} \frac{A^+(z)}{z^{n+1}}\,dz + \frac{1}{2\pi i} \int_{|z|=\rho_2} \frac{A^-(z)}{z^{n+1}}\,dz \qquad (\rho_1 < R,\ \rho_2 > r).$$

From the elementary properties of Laurent expansions there follow the following theorems.

Theorem 1. If $a_n \to A(z)$, $b_n \to B(z)$, and there exists an annulus in which $A(z)$ and $B(z)$ are simultaneously regular functions, then

$$\sum_{k=-\infty}^{\infty} a_k b_{n-k} \ \to\ A(z)B(z).$$

This theorem is analogous to the convolution formula in the theory of Fourier transforms.

Theorem 2. Let $A(z)$ be regular in the annulus $r_a < |z| < R_a$, and $B(z)$ regular in the annulus $r_b < |z| < R_b$ $(r_b > R_a)$, and let, for all $n = \ldots, -1, 0, 1, \ldots$, the equalities

$$\frac{1}{2\pi i} \int_{|z|=\rho_1} \frac{A(z)\,dz}{z^{n+1}} = \frac{1}{2\pi i} \int_{|z|=\rho_2} \frac{B(z)\,dz}{z^{n+1}} \qquad (r_a < \rho_1 < R_a,\ r_b < \rho_2 < R_b)$$

hold.

Under these conditions $A(z)$ and $B(z)$ are analytically continued into the annulus $r_a < |z| < R_b$, and in this annulus $A(z) = B(z)$.

This theorem is analogous to the "general theorem" ((1), p. 329), applied in the theory of Fourier transforms for passing from integral equations of convolution type to problems of the theory of functions of a complex variable.

§ 2. Consider the homogeneous system (1), assuming that a_n possesses the transform $A(z)$, regular in the annulus $r < |z| < R$, and we shall seek a solution x_n satisfying the conditions

$$\left[\lim_{n\to\infty} \sqrt[n]{|x_n|}\right]^{-1} = R_0 > r, \qquad \lim_{n\to\infty} \sqrt[n]{|x_{-n}|} = r_0 < R.$$

Proceeding analogously to the way this was done in (1) in solving the integral equation (3), and using Theorems 1 and 2 in place of the convolution formula and the "general theorem," we obtain that the unknowns x_k will be the coefficients of the expansions of the functions

$$X^+(z) = \sum_{k=0}^{\infty} x_k z^k \quad \text{and} \quad X^-(z) = \sum_{k=-\infty}^{-1} x_k z^k,$$

which, in the annulus $r < |z| < R$, which we shall denote by D, satisfy the conditions:

1) $X^+(z)[1 - A(z)]$, $X^-(z)[1 - A(z)]$ are regular in D;
2) $X^+(z)[1 - A(z)] = -X^-(z)[1 - A(z)]$ in D.

Thus, $X^+(z)$ and $X^-(z)$ are regular in D everywhere except, possibly, at the zeros of the function $1 - A(z)$, and in this annulus $X^+(z) = -X^-(z)$. The unknowns x_n are now computed as follows:

$$x_n = \frac{1}{2\pi i} \int_{|z|=r+\varepsilon} \frac{X^+(z)}{z^{n+1}}\, dz + \frac{1}{2\pi i} \int_{|z|=R-\varepsilon} \frac{X^-(z)}{z^{n+1}}\, dz =$$

$$= \frac{1}{2\pi i} \int_L \frac{X^+(z)}{z^{n+1}}\, dz = \frac{1}{2\pi i} \int_L \frac{X^+(z)(1 - A(z))}{z^{n+1}} \frac{dz}{1 - A(z)} =$$

$$= \frac{1}{2\pi i} \int_L \frac{X^+(z)(1 - A(z))}{z^{n+1}} \sum_{k=1}^{N} \sum_{p=1}^{\nu_k} \frac{A_{kp}\, dz}{(z - z_k)^p}.$$

Here L consists of two circles: $|z| = r + \varepsilon$ and $|z| = R - \varepsilon$, and ε is sufficiently small.

Thus, for x_n one obtains the expression

$$x_n = -\frac{c_1}{z_k^{n+1}} + \frac{c_2(n+1)}{z_k^{n+2}} + \cdots + \frac{c_{\nu_k}(n+1)\cdots(n+\nu_k-1)}{z_k^{n+\nu_k}}.$$

Here ν_k is the multiplicity of the zero z_k of the function $1 - A(z)$; N is the number of these zeros in D.

A particular solution of the nonhomogeneous system (1) is found by passing from system (1) to its Laurent transform:

$$X(z) = A(z)X(z) + B(z)$$

$$\left(B(z) = \sum_{k=-\infty}^{\infty} b_k z^k \right).$$

The unknowns x_k will be found as the coefficients of the Laurent expansion of the function

$$X(z) = \frac{B(z)}{1 - A(z)}$$

in some annulus with center at $z = 0$, not containing zeros of the function $1 - A(z)$.

§ 3. We shall solve system (2) by analogy with the known methods of V. A. Fock (^{2}) and Hopf–Wiener (^{3}), applied in the theory of integral equations (^{4}).

First consider the homogeneous system (2). Suppose that

$$A(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

is regular for $r < |z| < R$. We shall seek a solution such that

$$\left[\overline{\lim} \sqrt[n]{|x_n|}\right]^{-1} = R_0 > r.$$

Using Theorems 1 and 2, by analogy with the method proposed in (^{2,3}), we arrive at the following problem.

Find functions $X^+(z)$ and $X^-(z)$, the first of which is regular for $|z| < R$, except for zeros of the function $A(z) - 1$, where it may have poles of order not exceeding the order of the corresponding zero of the function $A(z) - 1$, while the second is regular for $|z| > r$ and tends to zero at infinity, if in the annulus D the relation

$$X^+(z)[A(z) - 1] = X^-(z)$$

holds.

Following F. D. Gakhov and Yu. I. Cherskii (^{4}), we shall call such a problem a plane problem.

It can be shown that the totality of solutions of this problem coincides with the solutions of the Riemann boundary-value problem* $X^+(t)[A(t)-1] = X^-(t)$, whose condition is prescribed on the circle $|z| = r - \varepsilon$, where ε is so small that in the annulus $r < |z| < r + \varepsilon$ there are no zeros of the function $A(z) - 1$. Nor is it difficult to solve the plane problem directly by a method almost literally coinciding with the method of solving the Riemann problem. It turns out that the totality of solutions of the plane problem indeed contains such functions $X^+(z)$ as have poles at the zeros of the difference $A(z) - 1$.

In solving the nonhomogeneous system, with respect to the free terms b_n one should assume that

$$\left[\overline{\lim}_{n\to\infty} \sqrt[n]{|b_n|}\right]^{-1} = R_1 > R.$$

Analogously to the case of the homogeneous system, one can obtain the plane problem

$$X^+(z)[A(z)-1] - X^-(z) = -B(z) \quad (r < |z| < R),$$

where $X^+(z)$, $X^-(z)$, $A(z)$ have the same meaning as before, and

$$B(z) = \sum_{n=0}^{\infty} b_n z^n.$$

The solution of this problem can also be carried out either by relying on the solution of the Riemann problem, or directly.

* In article (^{6}) this problem is called the Hilbert problem; however, this name is historically not justified (^{5}).

System (2), written somewhat differently, was considered in the work (6); however, as we shall show, a complete solution was not obtained in that work. It has already been mentioned above that the planar problem corresponding to system (2) is equivalent to the Riemann problem under a suitable choice of contour. If, however, as the contour one takes a circle lying in the annulus $r < |z| < R$ and containing within it the zeros of the function $A(z) - 1$ ($G(z)$ in the notation of Ya. N. Fel' d), then part of the solutions of the planar problem, and consequently part of the solutions of system (2), will be lost. Namely, the solutions of system (2) corresponding to solutions $X^+(z)$ having poles at zeros of the function $A(z) - 1$ that belong to the interior of the circle carrying the boundary condition will be lost.

§ 4. The integro-differential equation of infinite order

$$\sum \alpha_k y^{(k)}(z) = \Phi(z) \qquad \left(y^{(k)}(z) = \int_0^z y^{(k+1)}(t)\,dt \quad (k \leq 0), \qquad y^{(0)} = y \right),$$

if its solutions are sought in the class of entire functions of the first order of normal type

$$y = \sum_{k=0}^{\infty} \frac{x_k z^k}{k!},$$

is reduced to system (2). Here it is assumed that $\Phi(z)$ is also an entire function of the first order of normal type. The difference equation with constant coefficients:

$$\alpha_s f(t+s) + \alpha_{s-1} f(t+s-1) + \cdots + \alpha_0 f(t) = b(t) \qquad (t = 0, 1, 2, \ldots)$$

is a special case of system (2). In this case $A(z)$ is a polynomial.

In conclusion, we note that the method used in the present communication makes it possible to consider a further series of types of infinite systems of linear algebraic equations, by analogy with other integral equations of convolution type. In particular, "paired" infinite systems of linear algebraic equations are amenable to investigation by this method,

$$x_n = \sum_{k=-\infty}^{\infty} a_{n-k} x_k + b_n \qquad (n = 0, 1, 2, \ldots),$$

$$x_n = \sum_{k=-\infty}^{\infty} \tilde{a}_{n-k} x_k + b_n \qquad (n = -1, -2, \ldots),$$

analogous to paired integral equations of convolution type considered in the work (4).

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Note: Figure translations are in progress. See original paper for figures.

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