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Abstract

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THEORY OF ELASTICITY

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ON INVERSE BOUNDARY-VALUE PROBLEMS OF THE NONLINEAR THEORY OF SHALLOW SHELLS

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Let the middle surface S^0 of a thin shallow shell of constant thickness t, before application of the load, be obtainable by normal displacements w^0 from a certain surface of regular form S, called the reference system; $\alpha = $ const and $\beta = $ const are the lines of curvature of the surface S; k_{11} and k_{22} are its principal curvatures; A and B are the coefficients of the first quadratic form; w is the deflection function of the points of S^0 under the action of the load, carrying S^0 into the position S^* after deformation. Then the determination of the relation between the external loads and the deformed state of the shell is reduced ([1]) to a boundary-value problem for a system of two differential equations with respect to w and the stress function ψ:

$$\Delta\Delta\psi + Et(\Gamma^* - \Gamma^0) = 0; \tag{1}$$

$$D\Delta\Delta w + T_{11}k^*_{11} + 2T_{12}k^*_{12} + T_{22}k^*_{22} - p = 0, \tag{2}$$

where E is the modulus of elasticity; $k^*_{ij} = k_{ij} + \chi^0_{ij} + \chi_{ij}$;
$\chi^0_{11} = -\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w^0}{\partial\alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w^0}{\partial\beta}, \dots,$ χ_{12} are the corresponding changes of curvature;
$\Gamma^* = -k^{*2}_{12} + k^*_{11}k^*_{22}$; $\Gamma^0 = -\chi^{02}_{12} + (k_{11} + \chi^0_{11})(k_{22} + \chi^0_{22})$ are the Gaussian curvatures of the surfaces S^* and S^0; Δ is the Laplace operator; $D = Et^3/[12(1-\nu^2)]$;

$$T_{11} = \frac{1}{B}\left[\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial\psi}{\partial\beta}\right) + \frac{1}{A^2}\frac{\partial B}{\partial\alpha}\frac{\partial\psi}{\partial\alpha}\right], \tag{3}$$

$$T_{12} = -\frac{1}{AB}\frac{\partial^2\psi}{\partial\alpha\,\partial\beta} + \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial\psi}{\partial\beta} + \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial\psi}{\partial\alpha}$$

are membrane stresses; $p > 0$ is internal pressure; $w > 0$ if the displacement is directed along the outward normal.

In the direct problem of shell theory all quantities except w and ψ are considered prescribed; moreover, equations (1) and (2) are both nonlinear. One may pose the following inverse problems of the nonlinear theory of shells, which are of considerable interest.

A. Suppose that, for a given reference system, it is required to find such a form of the surface S^0 before deformation that, under the action of a prescribed pressure p and prescribed contour forces and geometric boundary conditions, the shell assumes a prescribed form.

In this case, since $w^* = w^0 + w$ and k^*_{ij} are prescribed quantities, equation (2), according to formulas (3), is linear with respect to the unknowns w^0 and ψ, while equation (1) remains nonlinear. Here the total order of equations (1) and (2) is equal to 8, as in the direct problem, which makes it possible to satisfy the usual boundary conditions of the theory of thin shells.

If, moreover, Γ^0 is small in comparison with Γ^*, then in a first approximation equation (1) can also be linearized by neglecting in it the term containing Γ^0.

B. Let the reference system and the functions w^0 and w, satisfying the corresponding boundary conditions, be given, and let it be required to find the pressure and the stress function ψ. Then from the linear equation (1) we determine ψ, substitution of which in (2) directly gives the desired pressure.

We note that the solutions of these inverse problems can also be used to find an approximate form of the deflection suitable for solving the direct problem by one of the approximate methods.

As an example of inverse problems of type A, consider the symmetric deformation of a shallow segment of a shell of revolution, hinged to a rigid circular contour of radius a. We take as the reference system the polar coordinate system (r, θ) in the plane S. Then $A = 1,\ B = r$,

$$\chi_1 = -\frac{d^2 w}{dr^2}, \qquad \chi_2 = -\frac{1}{r}\frac{dw}{dr},$$

$$T_{11} = \frac{1}{r}\frac{d\psi}{dr}, \qquad T_{22} = \frac{d^2\psi}{dr^2}, \qquad \Delta(\ldots) = \frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}(\ldots)\right]. \tag{4}$$

Introducing these quantities into (1) and (2), and also using the notation

$$w^* = w + w^0, \qquad \rho = \frac{r^2}{a^2}, \qquad \frac{1}{r}\frac{dw^*}{dr} = \omega^*, \qquad \frac{1}{r}\frac{dw^0}{dr} = \omega^0, \tag{5}$$

we obtain equations that can be integrated once; moreover, the arbitrary constants of integration should be set equal to zero if the shell has no opening at the pole.

Thus we obtain the equations

$$\frac{8}{Eta^2}\frac{d^2}{d\rho^2}(\rho T_{11}) + \omega^{*2} - \omega^{02} = 0; \tag{6}$$

$$\frac{4D}{a^2}\frac{d^2}{d\rho^2}(\rho\omega^* - \rho\omega^0) - T_{11}\omega^* - \frac{1}{2\rho}\int p\,d\rho = 0. \tag{7}$$

Consider the particular case when

$$p = \text{const}, \qquad a\omega^* = c = \text{const}. \tag{8}$$

In this case

$$w^* = \frac{cr^2}{2a}, \qquad \frac{d^2w^*}{dr^2} = \frac{c}{a} = \frac{1}{r}\frac{dw^0}{dr},$$

i.e., after deformation the shell becomes spherical.

We determine the shape of the shell before deformation, characterized by the initial deviation w^0 from the plane S.

From (7) and (6) we find

$$\frac{c}{a}T_{11} = -\frac{p}{2} - \frac{4D}{a^2}\frac{d^2}{d\rho^2}(\rho\omega^0); \tag{9}$$

$$\omega^{02} = \frac{c^2}{a^2} - \frac{8t^2}{3(1-\nu^2)a^3c}\frac{d^2}{d\rho^2}\left[\rho\frac{d^2}{d\rho^2}(\rho\omega^0)\right]. \tag{10}$$

We seek the solution of equation (10) in the form of a series

$$\omega^0 = \frac{c}{a}(b_0 + b_1\rho + b_2\rho^2 + ...). \tag{11}$$

Introducing the notation

$$\eta = (1-\nu^2)a^2c^2/32t^2, \tag{12}$$

we obtain recurrence formulas that allow $b_2, b_3, ...$ to be expressed in terms of b_0 and b_1:

$$b_2 = (1 - b_0^2)\eta, \qquad b_3 = -\frac{1}{3} b_0 b_1 \eta,$$

$$b_{2n+2} = -3\eta \left[b_n^2 + 2\sum_{i=0}^{n-1} b_i b_{2n-i} \right] : \left[(2n+3)(n+1)^2(2n+1) \right], \tag{13}$$

$$b_{2n+3} = -6\eta \left[\sum_{i=0}^{n} b_i b_{2n+1-i} \right] : \left[(n+2)(2n+3)^2(n+1) \right], \qquad n \geq 1.$$

The radial displacement on the contour is equal to zero if the condition is satisfied

$$T_{22} - \nu T_{11} = 0 \qquad \text{for } \rho = 1,$$

which, according to (4), (9), and (11), is reduced to the form

$$q = -\frac{3p(1-\nu)(1-\nu^2)a^3}{2Et^3 c} = \sum_{n=1}^{\infty} n(n+1)(2n-1-\nu)b_n. \tag{14}$$

In addition, for a hinged fixing the condition

$$\frac{d^2 w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr} = 0 \qquad \text{for } r = a,$$

must be satisfied, which, taking into account (5), (8), and (11), can be brought to the form

$$\sum_{n=1}^{\infty} (2n+1+\nu)b_n = (1+\nu)(1-b_0). \tag{15}$$

For given η and q, from (14) and (15), taking into account formulas (13), we obtain two equations with respect to the quantities b_0 and b_1; moreover, the convergence of the series entering equation (14) must be ensured, since the series (11) and (15) converge more rapidly.

For $\eta < 1$ these series converge fairly rapidly, but nevertheless the computations necessary for a direct determination of b_0 and b_1 prove difficult to carry out because of the nonlinearity of the corresponding equations. It is much simpler to proceed in the inverse way: for a given η, assigning values of b_0, beginning with a value close to unity, and determining b_1 from (15) and (13), with the corresponding values of q being determined by formula (14). In this way it was found, for example, that for $\eta = 0.5$ and $b_0 = 0.8$ we have $b_1 \approx -0.265$ and $q \approx +3.554$. This case corresponds to internal pressure, since $c < 0$.

We have also considered the solution of several other inverse problems of the nonlinear theory of shells, which we do not discuss here.

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REFERENCES

1. Kh. M. Mushtari, *Prikladn. matem. i mekh.*, **15**, issue 6 (1951).

Note: Figure translations are in progress. See original paper for figures.