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Abstract

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ON FUNCTIONS CONTINUOUS IN EACH VARIABLE

(Presented by Academician A. N. Kolmogorov on 14 IX 1956)

W. Sierpiński, and then independently of him G. P. Tolstov, proved the following theorem ([1],[2]):

Theorem. *A real function of two real variables, continuous in each variable in a domain G, is completely determined by its values at the points of a set E everywhere dense in G.*

In connection with this theorem certain questions arise:

A. Continuous functions possess a stronger property: if two real functions coincide on a set E everywhere dense in a domain G and are continuous at each point of $G - E$, then they coincide in G. Is this property preserved if, instead of continuity, one requires only continuity in each variable?

B. G. P. Tolstov's proof of this theorem, as well as the proofs (with one exception) given by W. Sierpiński, use the property of functions of two variables, continuous in each variable, of belonging to the first Baire class.

But a function of n variables, continuous in each variable, is a function of class $\leq n-1$ and, as Lebesgue showed ([3]), may be effectively of class $n-1$. As for the proof in ([1]) that does not use this property, it can also be carried out for functions of n variables ($n > 2$), but it becomes more complicated as n grows. Is it possible to give a simple proof of the Sierpiński–Tolstov theorem that does not depend on the number of variables?

In the present note we answer these questions.

Theorem 1. *There exists a function $f(x, y)$, defined in the unit square P, equal to zero on a set E everywhere dense in P, discontinuous in each variable at every point of $P - E$, and not identically equal to zero in P.*

Proof. The set $E \subset P$ is defined as follows: $(x, y) \in E$ if there exists a natural number p such that

$$x = \sum_{n=1}^{p} \frac{\varepsilon_n}{2^n}, \qquad \varepsilon_n = \begin{cases} 0 \text{ or } 1, & \text{if } n < p, \\ 1, & \text{if } n = p; \end{cases}$$

$$y = \sum_{n=1}^{p} \frac{\eta_n}{2^n}, \qquad \eta_n = \begin{cases} 0 \text{ or } 1, & \text{if } n < p, \\ 1, & \text{if } n = p. \end{cases}$$

It is easy to see that E is everywhere dense in P and has a finite number of points on every line parallel to one of the coordinate axes. The characteristic function of the set $P - E$ satisfies the condition of the theorem.

* We take this opportunity to note that the corollary to Theorem III in ([4]) is erroneous.

Theorem 2. Let $f(x, y)$ be defined in a domain G and satisfy the following conditions:

a) $f(x, y)$ is continuous in x at every point of the domain G;

b) there exists $E \subset G$, everywhere dense in G, such that $f(x, y)$ vanishes on E;

c) $f(x, y)$ is continuous in y at every point of $G - E$.

Then $f(x, y) = 0$ in G.

Proof. Suppose that there exists a point $(x_0, y_0) \in G$ such that $f(x_0, y_0) \neq 0$. For definiteness let $f(x_0, y_0) > 0$.

Let γ be such that $0 < \gamma < f(x_0, y_0)$. By a) there exists $\eta > 0$ such that $f(x, y_0) > \gamma$ for $x \in (x_0 - \eta,\ x_0 + \eta)$. By b), to each $x \in (x_0 - \eta,\ x_0 + \eta)$ there corresponds an $\varepsilon_x > 0$ such that $f(x, y) \geqslant \gamma$ for $y \in (y_0 - \varepsilon_x,\ y_0 + \varepsilon_x)$.

Let A_n be the set of points (x, y_0) such that $x \in (x_0 - \eta,\ x_0 + \eta)$ and $\varepsilon_x \geqslant 1/n$.

There exist a natural number p and a subinterval I of the interval $(x_0 - \eta,\ x_0 + \eta)$ such that A_p is everywhere dense in I. Indeed, otherwise A_n would be nowhere dense in $(x_0 - \eta,\ x_0 + \eta)$ for all n, and, since

$$(x_0 - \eta,\ x_0 + \eta) = \bigcup_{n=1}^{\infty} A_n,$$

we arrive at a contradiction.

Consider the rectangle D, which is the Cartesian product of the intervals I and $(y_0 - 1/p,\ y_0 + 1/p)$. By b) there exists inside D a point (x_1, y_1) such that $f(x_1, y_1) = 0$. By a) there exists $\omega > 0$ such that $f(x, y_1) < \gamma$ for $x \in (x_1 - \omega,\ x_1 + \omega)$.

Let $x \in A_p \cap (x_1 - \omega,\ x_1 + \omega)$; since $y_1 \in (y_0 - 1/p,\ y_0 + 1/p)$, we have $f(x, y_1) \geqslant \gamma$.

The contradiction obtained proves the theorem.

Theorem 3. Let $f(x,y)$ be defined in a domain G and satisfy the following conditions:

a) there exists a residual set $E \subset G$ such that $f(x,y)=0$ for $(x,y)\in E$;

b) $f(x,y)$ is continuous in each variable in $G-E$.

Then $f(x,y)=0$ in G.

Proof. Suppose that there exists a point $(x_0,y_0)\in G$ such that $f(x_0,y_0)\neq 0$. Let, for definiteness, $f(x_0,y_0)>0$. One can construct a rectangle D as in the preceding theorem.

Since E is residual in G, E is residual also relative to $D\subset G$.

Let $y=\lambda$ be a straight line whose intersection with E is a linear residual set T (see ([5]), Lemma 6); here λ is chosen in the interval $(y_0-1/p,\ y_0+1/p)$.

Let (ξ,λ) be such that $f(\xi,\lambda)\geqslant\gamma>0$ (it exists by the construction of the rectangle D). Since $(\xi,\lambda)\in G-E$, by b) $f(x,y)$ is continuous in x at the point (ξ,λ). But arbitrarily close to (ξ,λ) on the line $y=\lambda$ there exist points of $T\subset E$, i.e. points at which, by a), the function vanishes.

The contradiction obtained proves the theorem.

Theorem 4. If real functions $f(x_1,\dots,x_n)$, $g(x_1,\dots,x_n)$ of real variables, defined in D, coincide on a set E everywhere dense in D, and if they are continuous in each variable at all points of D, then they coincide in D.

Proof. Let

$$\varphi(x_1,\dots,x_n)=f(x_1,\dots,x_n)-g(x_1,\dots,x_n).$$

The function $\varphi(x_1,\dots,x_n)$ vanishes on E and is continuous in D in each variable. By the theorem of S. Kempisty ([6]) it follows that $\varphi(x_1,\dots,x_n)$ is quasicontinuous in D. Suppose that there exists a point $P_0\in D$ such that $\varphi(P_0)\neq 0$. For every $\varepsilon>0$ there exists an n-dimensional interval $I\subset D$ such that for $P\in I$ we have $|\varphi(P)-\varphi(P_0)|<\varepsilon$ (by the definition of quasi-continuity). Let $\varepsilon<|\varphi(P_0)|$. Since $\varphi(x_1,\dots,x_n)$ vanishes on E, there exists a point P_1 in I such that $\varphi(P_1)=0$; consequently, $|\varphi(P_0)|<\varepsilon$.

The contradiction obtained proves the theorem.

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Note: Figure translations are in progress. See original paper for figures.