Mathematics

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Abstract

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On the Computation of Eigenvalues of an Integral Equation by Means of Traces of Iterated Kernels

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Let $K(s,t)$ be a real, symmetric, and continuous kernel in the square $a \leq s, t \leq b$. For simplicity we shall assume the kernel to be positive in the sense of integral equations. We shall denote the traces of the iterated kernels by

$$k_j = \int_a^b K_j(s,s)\,ds, \qquad j = 1, 2, \ldots$$

It is known that the logarithmic derivative of the Fredholm determinant $D(\lambda)$ of the kernel $K(s,t)$ is represented by the power series

$$\frac{D'(\lambda)}{D(\lambda)} = -k_1 - k_2\lambda - k_2\lambda^2 < \cdots, \tag{1}$$

convergent for $|\lambda| < \lambda_1$, where λ_1 is the smallest eigenvalue of the kernel $K(s,t)$.

We shall consider the analytic function $D(\lambda)/D'(\lambda)$ and its expansion in a power series

$$\frac{D(\lambda)}{D'(\lambda)} = a_1 + a_2\lambda + a_3\lambda^2 + \cdots \tag{2}$$

The radius of convergence of the series on the right-hand side of (2), λ_1', is greater than λ_1. This follows from the following easily proved proposition: the roots of the derivative of the Fredholm determinant of a symmetric kernel are real; if the kernel is positive, then none of the roots of $D'(\lambda) = 0$ can lie to the left of λ_1.

The fact that the radius of convergence of the series (2) is greater than λ_1 makes it possible to find λ_1 as a root of the equation

$$f(\lambda) \equiv a_1 + a_2\lambda + a_3\lambda^2 + \cdots = 0. \tag{3}$$

The series (2) can be obtained as the result of dividing unity by the series (1); in this case the coefficients a_k are determined by the recurrence relations

$$k_1 a_1 = -1; \qquad k_1 a_2 + k_2 a_1 = 0; \qquad k_1 a_3 + k_2 a_2 + k_3 a_1 = 0; \ldots \tag{4}$$

The principal result used in what follows is that the coefficients a_k satisfy the inequalities

$$a_k a_{k+2} - a_{k+1}^2 > 0 \qquad (k \geq 2). \tag{5}$$

In particular, from this and from the fact that a_2 and $a_3 > 0$, it follows that $a_k > 0$ for $k \geq 2$.

Let the traces of the iterated kernels $k_1, k_2, \ldots, k_m$ be known. By means of relations (4) one can compute $a_1, a_2, \ldots, a_m$. Assuming $m \geq 3$, introduce the notation $q = a_m / a_{m-1}$. From inequalities (5) it follows that

$$a_{m+1} > q a_m; \qquad a_{m+2} > q^2 a_m; \ldots \tag{6}$$

Along with equation (3), consider the equation

$$S_m(\lambda) = a_1 + a_2\lambda + \cdots + a_m \lambda^{m-1} + \lambda^m q a_m (1 + \lambda q + \lambda^2 q^2 + \cdots) = 0$$

or

$$S_m(\lambda) = a_1 + a_2\lambda + \cdots + a_{m-2}\lambda^{m-3} + \frac{\lambda^{m-2} a_{m-1}^2}{a_{m-1} - \lambda a_m} = 0. \tag{7}$$

The smallest positive root $\lambda^{(m)}$ of this equation is taken as an approximation to λ_1. It is clear that $\lambda^{(m)} > \lambda_1$, since, by virtue of (6), $f(\lambda) > S_m(\lambda)$ for $\lambda > 0$.

The well-known trace method (1) makes it possible to indicate approximations from below and from above for λ_1:

$$\sqrt[m]{\frac{p_1}{k_m}} < \lambda_1 < \frac{k_{m-1}}{k_m}. \tag{8}$$

Here p_1 is the unknown multiplicity of λ_1. It can be proved that for sufficiently large m

$$\lambda^{(m)} < k_{m-1}/k_m,$$

i.e., $\lambda^{(m)}$ approximates λ_1 more accurately than k_{m-1}/k_m.

Equation (7) is naturally solved by Newton' s method, taking k_{m-1}/k_m as the initial approximation to its root. In this process one can determine the multiplicity p_1. Indeed, the following approximate equality holds:

$$\frac{1}{p_1} = S'_m\left(\frac{k_{m-1}}{k_m}\right).$$

Knowing p_1, we can also indicate the lower bound in (8).

We note that $S_3(k_2/k_3) = 0$, so that the proposed method is meaningful to apply for $m \geq 4$.

The case $m = 2$, which is practically the most important, does not fit into the scheme given above, and we shall consider it separately. Thus, let k_1 and k_2 be known. From inequality (2)

$$k_1^3 - 3k_1k_2 + 2k_3 = \int_a^b\int_a^b\int_a^b K\begin{pmatrix}t_1, t_2, t_3\\ t_1, t_2, t_3\end{pmatrix} dt_1 dt_2 dt_3 \geq 0$$

we find

$$k_3 \geq \frac{1}{2}(3k_1k_2 - k_1^3).$$

Since, by (4), $a_3 = k_1^{-3}(k_1k_3 - k_2^2)$, it follows that

$$a_3 \geq k_1^{-3}\left[k_1 \cdot \frac{1}{2}(3k_1k_2 - k_1^3) - k_2^2\right] = k_1^{-3}(k_1^2 - k_2)(k_2 - \tfrac{1}{2}k_1^2) = a_3^*. \qquad (9)$$

Obviously, $k_1^2 - k_2 \geq 0$, and the equality sign occurs if and only if the kernel $K(s,t)$ has a single simple eigenvalue. Excluding this uninteresting case, we obtain for a_3 the positive lower bound a_3^*, if

$$\frac{k_2}{k_1^2} > \frac{1}{2}. \qquad (10)$$

Suppose that condition (10) is satisfied. Then $a_3 \geqslant a_3^* > 0$. Denote $q^* = a_3^*/a_2$. Obviously, $q^* \leqslant q = a_3/a_2$. From inequalities (6) it follows that

$$a_4 > q^*a_3^*; \qquad a_5 > q^*a_3^*; \ldots$$

Let us form the equation

$$S_3^*(\lambda) = a_1 + a_2\lambda + a_2^*\lambda^2 + q^*a_3^*\lambda^3 + q^*a_3^*\lambda^4 + \ldots = 0.$$

By virtue of the preceding inequalities, for $\lambda > 0$,

$$f(\lambda) > S_3^*(\lambda) > a_1 + a_2\lambda,$$

therefore the root of the equation $S_3^*(\lambda) = 0$, equal to

$$\lambda^{(2)*} = \frac{2k_2}{k_1(3k_2 - k_1^2)}, \tag{11}$$

is an upper bound for λ_1, more accurate than k_1/k_2.

Condition (10) is not always satisfied. Let us indicate one case in which this condition is satisfied. Suppose that the traces k_r and k_{2r} are known to us, where r is a positive integer. Such a situation occurs, for example, when the method of traces is applied, computing k_p for $p = 1, 2, 4, 8, \ldots$. Considering $K_r(s,t) = L(s,t)$ as the initial kernel and denoting the traces of its iterated kernels by $l_1, l_2, l_3, \ldots$, we have $l_1 = k_r,\ l_2 = k_{2r},\ l_3 = k_{3r}$, etc.

From the relation

$$\frac{l_2}{l_1^2} = \frac{k_{2r}}{k_r^2} = \left(\frac{p_1}{\lambda_1^{2r}} + \frac{p_2}{\lambda_2^{2r}} + \ldots\right)\left(\frac{p_1}{\lambda_1^{r}} + \frac{p_2}{\lambda_2^{r}} + \ldots\right)^{-2}$$

it is seen that, for sufficiently large r, the ratio l_2/l_1^2 will be close to $1/p_1$, and condition (10) for $L(s,t)$ proves to be satisfied if λ_1 is a simple eigenvalue ($p_1 = 1$). To find p_1, it is sufficient to calculate the ratio k_r^2/k_{2r}.

In the case of arbitrary multiplicity p_1 one can prove that, for sufficiently large r, the inequalities

$$\lambda_1^r < 2k_{2r}\left(k_r - \frac{p_1 - 1}{\sqrt{p_1}}\sqrt{k_{2r}}\right)^{-1}\left[3k_{2r} - p_1\left(k_r - \frac{p_1 - 1}{\sqrt{p_1}}\sqrt{k_{2r}}\right)^2\right]^{-1} < \frac{k_r}{k_{2r}}$$

hold.

Inequalities (8) with $m = 2$, when applied to the kernel $L(s,t)$, are written as follows:

$$\sqrt{\frac{p_1}{k_{2r}}} < \lambda_1^r < \frac{k_r}{k_{2r}}. \tag{12}$$

The significance of these inequalities is that they make it possible to give an error estimate for the approximate values found for λ_1. But, on the other hand, it is clear that the estimate obtained will be too large, since, obviously, the upper bound in (12) is cruder than the lower one. If the upper bound is replaced by the right-hand side of formula (11) as applied to the kernel $L(s,t)$ (for example, when $p_1 = 1$), then we obtain bounds of the same order of closeness to λ_1:

$$\sqrt{\frac{p_1}{k_{2r}}} < \lambda_1^r < \frac{2k_{2r}}{k_r(3k_{2r} - k_r^2)}. \tag{13}$$

Example. Consider the kernel

$$K(s,t) = \begin{cases} 10s(1-t), & \text{for } 0 \leqslant s \leqslant t \leqslant 1, \\ 10t(1-s), & \text{for } 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

We have $k_4 = 1.0582010$; $k_8 = 1.1107304$. Since $k_8/k_4^2 = 0.9919$, it follows that $p_1 = 1$.

Inequalities (12) give

$$0.9869596 < \lambda_1 < 0.9879612,$$

and for the error of the approximate value found we obtain the bound 10^{-3}.

Let us now use inequalities (13):

$$0.9869596 < \lambda_1 < 0.9869647.$$

Hence we obtain the error bound $0.51 \cdot 10^{-5}$.

The exact value is

$$\lambda_1 = 10^{-1}\pi^2 = 0.9869604\ldots.$$

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