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Abstract

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On the Theory of Hausdorff Transformations

(Presented by Academician A. N. Kolmogorov on 25 X 1956)

1. Transformations defined by triangular matrices

$$\|C_{m,n}\| \quad \text{and} \quad \|C_{m,m-n}\| \qquad (C_{m,k}=0 \text{ for } k>m \text{ and } k<0)$$

will be called conjugate transformations.

Theorem 1. *A transformation conjugate to a Hausdorff transformation is a Hausdorff transformation* ((1), p. 309).

The proof of this theorem may be based on the fact that Hausdorff transformations form, in the class of triangular transformations, a maximal commutative semigroup ((1), p. 310).

Theorem 2. *The sequences of complex numbers* $\{\mu_n\}$ *and* $\{\mu_n^*\}$ *defining conjugate Hausdorff transformations are connected with each other by a* δ*-transformation* ((1), p. 307).

Theorem 3. *If two sequences of complex numbers are connected with each other by a* δ*-transformation, then from the absolute monotonicity * ((1), p. 314) of one of these sequences there follows the absolute monotonicity of the other.*

Theorem 4. *In order that the conjugate Hausdorff transformations defined by the sequences of complex numbers* $\{\mu_n\}$ *and* $\{\mu_n^*\}$ *be regular transformations* ((1), p. 62), *it is necessary and sufficient that the following conditions be satisfied:*

A. The sequence $\{\mu_n\}$ is the difference of two absolutely monotone sequences.

B.

$$\sum_{n=0}^{m}(-1)^n\binom{m}{n}\mu_n \to 0 \quad \text{as } m\to\infty.$$

C. $\mu_n \to 0$ as $n\to\infty$.

D. $\mu_0 = 1$.

(The conditions remain valid if in them the numbers μ_n *are replaced by the numbers* μ_n^**.)*

Theorem 5. *In order that the complex functions $\chi(t)$ and $\chi^*(t)$, of bounded variation on the interval $[0,1]$, define conjugate regular Hausdorff transformations* ((1), p. 318), *it is necessary and sufficient that the following conditions be satisfied:*

A. $\chi^*(t) = 1 - \chi(1-t)$.

B. $\chi(+0) = 0$.

C. $\chi(1-0) = 1$.

D. $\chi(1) = 1$.

2. Let a sequence of complex numbers $\{p_n\}$ satisfy the condition:

$$p_0 + p_1 + \dots + p_n = P_n \neq 0$$

for all values of n. The conjugate transformations defined by the triangular matrices

$$\left\|\frac{p_n}{P_m}\right\| \quad \text{and} \quad \left\|\frac{p_{m-n}}{P_m}\right\|,$$

$n \leq m$, will be called Voronoi-Riesz transformations.

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* A sequence of complex numbers is absolutely monotone if the sequences of its real and imaginary parts are absolutely monotone.

The totality consisting of all Cesàro transformations ((1), p. 125) and of all transformations associated with them we call the class of transformations associated with Cesàro transformations.

Theorem 6. *In order that the Voronoi–Riesz transformation defined by a sequence of complex numbers $\{p_n\}$ belong to the class of Hausdorff transformations, it is necessary and sufficient that the numbers p_n satisfy the relation*

$$p_n = \frac{p_1}{p_0}\frac{P_{n-1}}{n}. \tag{1}$$

Theorem 7. *The intersection of the class of Hausdorff transformations with the class of Voronoi–Riesz transformations coincides with the class of transformations associated with Cesàro transformations.*

In the proof of Theorem 7 it is established that relation (1) is a necessary and sufficient condition for a Voronoi–Riesz transformation to belong to the class of transformations associated with Cesàro transformations.

3. Let a power series be given,

$$F(z) = \sum_{\lambda=1}^{\infty} \gamma_\lambda z^\lambda$$

and its powers in the sense of Cauchy

$$[F(z)]^\nu = \sum_{\lambda=\nu}^{\infty} \gamma_\lambda^{(\nu)} z^\lambda.$$

A transformation defined by the triangular matrix $\|\varphi_n(m) - \varphi_{n+1}(m)\|$ we call analytic if

$$\varphi_n(m) = \sum_{\lambda=n}^{m} \gamma_\lambda^{(n)}$$

and $\varphi_n(m) = 0$ for $n > m$. Analytic transformations differ insignificantly from the generalized Euler transformations studied by Perron ([2]).

If

$$F(z) = \frac{z}{1+q-qz},$$

then the analytic transformation defined by this function coincides with the Euler transformation of order q (([1]), p. 227). It is easy to show that transformations associated with Euler transformations are Euler transformations.

Theorem 8. *If a transformation associated with an analytic one is itself an analytic transformation, then it belongs to the class of Euler transformations.*

Theorem 9. *The intersection of the class of Hausdorff transformations with the class of analytic transformations coincides with the class of Euler transformations.*

In conclusion, let us note that regular Euler transformations and regular Cesàro transformations are contained in the class of associated regular Hausdorff transformations.

Ural State University
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CITED LITERATURE

[1] G. Hardy, *Divergent Series*, IL, 1951.
[2] O. Perron, *Math. Zs.*, **18**, 157 (1923).

Note: Figure translations are in progress. See original paper for figures.