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Abstract

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SEPARATED FLOW ACCORDING TO KIRCHHOFF' S SCHEME PAST ONE FAMILY OF CURVES IN A BOUNDED STREAM

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Closed solutions of the problem of separated flow past a prescribed obstacle by a stream of an ideal incompressible fluid, according to Kirchhoff' s scheme and bounded by two parallel walls, have been obtained only for a plate ([1−3]) and a wedge ([1]).

In the present work an exact solution of this problem is given in closed form for a certain two-parameter family of curves.

1. The curve defined by the equations

$$x = \frac{\lambda \cos^m \mu}{m \operatorname{tg} \mu} \int_\vartheta^{\frac{1}{2}\pi} \frac{\left(1+\sqrt{1-\operatorname{ctg}^2\mu \operatorname{tg}^2 \frac{\pi-2\vartheta}{2m}}\right) \cos^{-(m+2)\frac{\pi-2\vartheta}{2m}} \cos\vartheta \, d\vartheta}{\left(1+\operatorname{tg}^2\varepsilon \operatorname{ctg}^2\mu \operatorname{tg}^2 \frac{\pi-2\vartheta}{2m}\right)\left(1+\sin\mu\sqrt{1-\operatorname{ctg}^2\mu \operatorname{tg}^2 \frac{\pi-2\vartheta}{2m}}\right)^m}, \qquad (0 \le \mu < \pi/2m)$$

$$y = \frac{\lambda \cos^m \mu}{m \operatorname{tg} \mu} \int_\vartheta^{\frac{1}{2}\pi} \frac{\left(1+\sqrt{1-\operatorname{ctg}^2\mu \operatorname{tg}^2 \frac{\pi-2\vartheta}{2m}}\right) \cos^{-(m+2)\frac{\pi-2\vartheta}{2m}} \sin\vartheta \, d\vartheta}{\left(1+\operatorname{tg}^2\varepsilon \operatorname{ctg}^2\mu \operatorname{tg}^2 \frac{\pi-2\vartheta}{2m}\right)\left(1+\sin\mu\sqrt{1-\operatorname{ctg}^2\mu \operatorname{tg}^2 \frac{\pi-2\vartheta}{2m}}\right)^m},$$

where ϑ is the angle of inclination of the tangent to the x-axis, varying in the interval $0 \le \pi/2 - \vartheta \le m\mu$, will be called the curve $L(m, \mu, \varepsilon)$.

On passing to a new variable θ by means of the transformation $\vartheta = \pi/2 - m \operatorname{arctg}(\operatorname{tg}\mu \cos\theta)$, the equations of the curve $L(m, \mu, \varepsilon)$ are simplified and take the form:

$$x = \lambda \int_\theta^{\pi/2} \frac{(1+\sin\theta)\sin\theta(1-\sin^2\mu\sin^2\theta)^{m/2}}{(1+\operatorname{tg}^2\varepsilon\cos^2\theta)(1+\sin\mu\sin\theta)^m} \sin\left(m \arcsin\frac{\sin\mu\sin\theta}{\sqrt{1-\sin^2\mu\sin^2\theta}}\right) d\theta,$$

$$y = \lambda \int_{\theta}^{\pi/2} \frac{(1+\sin\theta)\sin\theta(1-\sin^2\mu\sin^2\theta)^{m/2}}{(1+\operatorname{tg}^2\varepsilon\cos^2\theta)(1+\sin\mu\sin\theta)^m} \cos\left(m\arccos\frac{\cos\mu}{\sqrt{1-\sin^2\mu\sin^2\theta}}\right) d\theta. \tag{1}$$

For the curvature $K(\vartheta)$ of the curve $L(m,\mu,\varepsilon)$ we have the expression

$$K(\vartheta) = -\frac{m\operatorname{tg}\mu}{\lambda\cos^m\mu} \frac{\left(1+\operatorname{tg}^2\varepsilon\operatorname{ctg}^2\mu\operatorname{tg}^2\frac{\pi-\vartheta}{2m}\right)\left(1+\sin\mu\sqrt{1-\operatorname{ctg}^2\mu\operatorname{tg}^2\frac{\pi-2\vartheta}{2m}}\right)^m}{\left(1+\sqrt{1-\operatorname{ctg}^2\mu\operatorname{tg}^2\frac{\pi-2\vartheta}{2m}}\right)\cos^{-(m+2)}\frac{\pi-2\vartheta}{2m}}. \tag{2}$$

The curve $L(m,\mu,\varepsilon)$ is symmetric with respect to the x-axis, passes through the origin, touching there the y-axis, and, for $0<\mu<\pi/2m$, is a monotonically increasing curve whose tangent at the endpoint C_1 forms with the x-axis an angle equal to $\pi/2-m\mu$ (Fig. 1a). For $\mu=0$, the curve $L(m,\mu,\varepsilon)$ becomes a straight line coinciding with the y-axis (Fig. 1b).

Let the length of the arc of the curve $L(m,\mu,\varepsilon)$ and the coordinates of the point C_1 be denoted, respectively, by $2S_0$, x_c, and y_c; then we shall have:

$$S_0 = \lambda D(m,\mu,\varepsilon), \qquad x_c = \lambda D_1(m,\mu,\varepsilon), \qquad y_c = \lambda D_2(m,\mu,\varepsilon); \tag{3}$$

$$D(m,\mu,\varepsilon) = \int_0^1 \frac{t(1+t)(1-t^2\sin^2\mu)^{m/2}\,dt}{\left[1+(1-t^2)\operatorname{tg}^2\varepsilon\right](1+t\sin\mu)^m\sqrt{1-t^2}};$$

$$D_1(m,\mu,\varepsilon) = \left|\int_0^{\pi/2} \frac{\sin\theta(1+\sin\theta)(1-\sin^2\mu\sin^2\theta)^{m/2}}{(1+\sin\mu\sin\theta)^m(1+\operatorname{tg}^2\varepsilon\cos^2\theta)} \sin\left(m\arcsin\frac{\sin\mu\sin\theta}{\sqrt{1-\sin^2\mu\sin^2\theta}}\right) d\theta\right|;$$

$$D_2(m,\mu,\varepsilon) = \left|\int_0^{\pi/2} \frac{\sin\theta(1+\sin\theta)(1-\sin^2\mu\sin^2\theta)^{m/2}}{(1+\sin\mu\sin\theta)^m(1+\operatorname{tg}^2\varepsilon\cos^2\theta)} \cos\left(m\arccos\frac{\cos\mu}{\sqrt{1-\sin^2\mu\sin^2\theta}}\right) d\theta\right|.$$

2. Consider the problem of separated flow past the curve $L(m,\mu,\varepsilon)$ by a plane flow of an ideal incompressible fluid according to Kirchhoff's scheme in a channel of width H, equal to

$$H = \pi\lambda\operatorname{cosec}^2\varepsilon\left(\frac{\cos\frac{\varepsilon+\mu}{2}}{\cos\frac{\varepsilon-\mu}{2}}\right)^m \operatorname{ctg}\frac{\pi-2\varepsilon}{4}. \tag{4}$$

Denote the velocity of the undisturbed flow at infinity and the velocity on the jets, respectively, by V_∞ and V_0. Let us take the arc $L(m, \mu, \varepsilon)$ being flowed around and the jets to be the streamline $\psi = 0$, and set the velocity potential φ at the points O and C_1 (Fig. 1) equal, respectively, to $\varphi = 0$ and $\varphi = \varphi_0$. The domain of the complex potential $w = \varphi + i\psi$, corresponding to the flow domain, will then be a strip of width $Q = HV_\infty$ (Fig. 1). By symmetry, it is possible to consider only the flow domain lying above the x-axis, to which in the w-plane there corresponds a strip of width $\frac{1}{2}Q$, lying above the φ-axis. To solve the problem it is sufficient to find the function

$$\chi(w) = \ln\left(\frac{dz}{dw}V_0\right), \tag{5}$$

analytic inside this strip and satisfying the conditions:

$$\chi(0) = \ln\infty = \infty; \qquad \chi(\infty) = 0;$$

$$\operatorname{Im}\chi = 0 \quad \text{for } \psi = 0,\ -\infty \le \varphi \le 0; \qquad \operatorname{Re}\chi = 0 \quad \text{for } \psi = 0,\ \varphi_0 \le \varphi \le \infty;$$

$$\operatorname{Im}\chi = 0 \quad \text{for } \psi = \frac{1}{2}Q,\ -\infty \le \varphi \le \infty;$$

$$K(\operatorname{Im}\chi)e^{\operatorname{Re}\chi} = V_0\frac{d}{d\varphi}(\operatorname{Im}\chi) \quad \text{for } \psi = 0,\ 0 \le \varphi \le \varphi_0.$$

Fig. 1: Schematic of separated flow in a channel.

It is easy to verify that such a function will be

$$\chi = \ln\left(\frac{1 - i\sqrt{\zeta^2 - 1}}{-i\zeta}\right)\left[\frac{1 - i(\zeta - \sqrt{\zeta^2 - 1})\operatorname{tg}\frac{\mu}{2}}{1 + i(\zeta - \sqrt{\zeta^2 - 1})\operatorname{tg}\frac{\mu}{2}}\right]^m, \tag{6}$$

$$\zeta = \sqrt{\exp(2\pi w/Q) - 1}\operatorname{ctg}\varepsilon, \qquad \cos\varepsilon = \exp(-\varphi_0\pi/Q).$$

In order that the function found satisfy the condition $\chi(-\infty) = \ln(V_0/V_\infty)$, it is necessary that the equality

$$V_0/V_\infty = \left(\cos\frac{\varepsilon + \mu}{2}\Big/\cos\frac{\varepsilon - \mu}{2}\right)^m \operatorname{ctg}\frac{\pi - 2\varepsilon}{4}, \tag{7}$$

hold, which is the equation for determining the magnitude V_0 from a given value of V_∞. Integrating (5) with allowance for (4), (6), and (7), we obtain the solution of the posed problem:

$$\frac{dw}{dz} = \frac{-iV_0\zeta}{1 - i\sqrt{\zeta^2 - 1}} \left[\frac{1 + i(\zeta - \sqrt{\zeta^2 - 1}) \operatorname{tg} \frac{\mu}{2}}{1 - i(\zeta - \sqrt{\zeta^2 - 1}) \operatorname{tg} \frac{\mu}{2}} \right]^m, \tag{8}$$

$$z = i\lambda \int_0^\zeta \frac{1 - i\sqrt{\zeta^2 - 1}}{1 + \zeta^2 \operatorname{tg}^2 \varepsilon} \left[\frac{1 - i(\zeta - \sqrt{\zeta^2 - 1}) \operatorname{tg} \frac{\mu}{2}}{1 + i(\zeta - \sqrt{\zeta^2 - 1}) \operatorname{tg} \frac{\mu}{2}} \right]^m d\zeta. \tag{9}$$

Passing in equality (8) to the limit as $\psi \to 0$, $\varphi = (\varphi_0 / \ln \sec^2 \varepsilon) \times \ln(1 + \operatorname{tg}^2 \varepsilon \operatorname{ch}^2 \theta_0)$, we find the equations of the jets

$$x_{\text{st}} = \lambda \left\{ D_1(m, \mu, \varepsilon) + \int_0^{\theta_0} \frac{\operatorname{ch} \theta_0 \operatorname{sh} \theta_0}{1 + \operatorname{tg}^2 \varepsilon \operatorname{ch}^2 \theta_0} \cos \left[m \arcsin \frac{(1 - \sin \mu) \operatorname{sh} \theta_0 + \cos \mu \operatorname{ch} \theta_0}{\operatorname{ch} \theta_0 (\cos \mu \operatorname{sh} \theta_0 + \operatorname{ch} \theta_0)} + (1 - m) \arcsin \frac{1}{\operatorname{ch} \theta_0} \right] \right. \tag{10}$$

$$y_{\text{st}} = \lambda \left\{ D_2(m, \mu, \varepsilon) + \int_0^{\theta_0} \frac{\operatorname{ch} \theta_0 \operatorname{sh} \theta_0}{1 + \operatorname{tg}^2 \varepsilon \operatorname{ch}^2 \theta_0} \sin \left[m \arcsin \frac{(1 - \sin \mu) \operatorname{sh} \theta_0 + \cos \mu \operatorname{ch} \theta_0}{\operatorname{ch} \theta_0 (\cos \mu \operatorname{sh} \theta_0 + \operatorname{ch} \theta_0)} + (1 - m) \arcsin \frac{1}{\operatorname{ch} \theta_0} \right] d \right.$$

and hence the curvature of the jet is

$$K_{\text{st}} = -\frac{(\cos \mu \operatorname{th} \theta_0 + 1 - m \sin \mu)(1 + \operatorname{tg}^2 \varepsilon \operatorname{ch}^2 \theta_0)}{\lambda \cos^2 \mu \operatorname{ch} \theta_0 \operatorname{sh} \theta_0 (1 + \cos \mu \operatorname{th} \theta_0)}. \tag{11}$$

The resistance of the arc $L(m, \mu, \varepsilon)$ is determined by the known formula

$$P = \left(1 - \frac{V_0}{V_\infty}\right)^2 \frac{\rho H V_\infty^2}{2}. \tag{12}$$

Let us note that, of the three parameters m, μ, and ε entering the solution, by virtue of relation (3) only two are arbitrary, independent of the width of the channel. If the channel width H, two of the quantities x_C, y_C, S_0, and the parameter m are specified, then, in order to determine the parameters μ and ε, according to (3), (4), we shall have two equations. We see that the resistance of the given arc $L(m, \mu, \varepsilon)$ and the velocity at any point of the flow are easily determined by formulas (12) and (8). The forms of the curve $L(m, \mu, \varepsilon)$ itself and of the jets are determined somewhat more complicatedly. However, for integer values of m the equation of the washed arc $L(m, \mu, \varepsilon)$ and the equation of the jets are expressed through elementary functions. As an example, let us consider the problem of separated flow past the curve $L(1, \mu, \varepsilon)$.

The equation of this curve, according to (1), has the form:

$$x = \frac{\lambda \sin\mu \cos^2\varepsilon}{\sin^2\mu - \sin^2\varepsilon} \left\{ \frac{\sin\mu - \sin^2\varepsilon}{2\sin^2\varepsilon} \ln(1 + \operatorname{tg}^2\varepsilon \cos^2\theta) + \frac{1 - \sin\mu}{\sin\mu} \left[\frac{\sin\mu}{\sin\varepsilon} (\operatorname{Artanh}(\sin\varepsilon \sin\theta) - \operatorname{Artanh}\sin\varepsilon) + \ln \right.\right.$$

$$y = \frac{\lambda \sin\mu \cos^2\varepsilon}{\sin^2\mu - \sin^2\varepsilon} \left\{ \frac{\sin\mu - \sin^2\varepsilon}{2\sin\varepsilon} \operatorname{arctg}(\operatorname{tg}\varepsilon \cos\theta) + (1 - \sin\mu) \left[\frac{\cos\varepsilon}{\cos\mu} \arcsin \frac{\cos\mu \cos\theta}{1 + \sin\mu \sin\theta} - \operatorname{arctg}(\sec\varepsilon \operatorname{ctg}\theta) \right] \right.$$

We find the solution of the problem by setting $m = 1$ in formulas (7)–(12); after transformations we obtain:

$$z(\zeta) = \frac{\lambda \cos^2\varepsilon}{\sin^2\mu - \sin^2\varepsilon} \left\{ \frac{\sin\mu - \sin^2\varepsilon}{\sin\varepsilon} \left[i\frac{\cos\mu}{\cos\varepsilon} \operatorname{arctg}(\zeta \operatorname{tg}\varepsilon) + \frac{\sin\mu}{2\sin\varepsilon} \ln(1 + \zeta^2 \operatorname{tg}^2\varepsilon) \right] + (1 - \sin\mu) \left[\frac{\cos\mu}{\cos\varepsilon} \ln \frac{\zeta +}{i\sqrt{1}} \right.\right.$$

the equation of the streamlines:

$$x_{\text{st}} = \lambda \left\{ D_1(1, \mu, \varepsilon) + \frac{\cos^2\varepsilon}{\sin^2\mu - \sin^2\varepsilon} \left[\frac{(1 - \sin\mu)\cos\mu}{\cos\varepsilon} \operatorname{Artanh}(\cos\varepsilon \operatorname{th}\theta_0) + \frac{\sin\mu - \sin^2\varepsilon}{2\sin^2\varepsilon} \ln(1 + \sin^2\varepsilon \operatorname{sh}^2\theta_0) \right.\right.$$

$$y_{\text{st}} = \lambda \left\{ D_2(1, \mu, \varepsilon) + \frac{\cos^2\varepsilon}{\sin^2\mu - \sin^2\varepsilon} \left[\frac{(\sin\mu - \sin^2\varepsilon)\cos\mu}{\sin\varepsilon \cos\varepsilon} (\operatorname{arctg}(\operatorname{tg}\varepsilon \operatorname{ch}\theta_0) - \varepsilon) + (1 - \sin\mu) \left(\frac{\sin\mu}{\sin\varepsilon} \operatorname{arctg}(\sin \right.\right.\right.$$

the drag:

$$P = \pi \left[1 - \left(\cos\frac{\mu + \varepsilon}{2} \Big/ \cos\frac{\mu - \varepsilon}{2} \right) \operatorname{ctg}\frac{\pi - 2\varepsilon}{4} \right]^2 \frac{\rho H V_\infty^2}{2}.$$

3. We have assumed that the parameters m and μ satisfy the condition $0 \leq \mu < \pi/2m$, under which the curve $L(m, \mu, \varepsilon)$ is monotonically increasing. If $\mu > \pi/2m$, then at the point $\theta = \arccos(\operatorname{ctg}\mu \operatorname{tg}(\pi/2m))$ the curve has a maximum; for $\mu = \pi/2m$ the curve $L(m, \mu, \varepsilon)$ has at the terminal point C_1 a tangent parallel to the x-axis. The streamlines for $\mu \geq \pi/2m$ have a different structure. For $\mu < \operatorname{arctg}\frac{2m}{m^2-1}$ the streamlines have an inflection point; for $\mu = \operatorname{arctg}\frac{2m}{m^2-1}$ the streamlines asymptotically approach the x-axis, while for

$$\mu > \operatorname{arctg} \frac{2m}{m^2 - 1}$$

they intersect it and pass onto the second sheet of the Riemann surface.

Earlier we obtained the solution of the problem of separated flow by an unbounded stream of an ideal incompressible fluid, according to the Kirchhoff scheme, past the curves $L(m, \mu)$ ([4]). The form of the curves $L(m, \mu)$ and of the streamlines changes with variation of the parameters m and μ in exactly the same way as for the curves $L(m, \mu, \varepsilon)$. It is easy to see that the limiting transition as $\varepsilon \to 0$ in the solution found above means the passage from the bounded stream flowing past the arc $L(m, \mu, \varepsilon)$ to the unbounded stream flowing past the arc $L(m, \mu)$.

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Note: Figure translations are in progress. See original paper for figures.