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Abstract

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ON DIFFERENTIAL EQUATIONS WITH UNBOUNDED OPERATORS IN HILBERT SPACE

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We shall consider the equation

$$\frac{dx}{dt} + A(t)x = f(t) \tag{1}$$

in a Hilbert space H, where $A(t)$, for each t $(0 \leq t \leq b)$, is a self-adjoint operator, and $f(t)$ is a function with values in H.

In the more general case of a Banach space, this equation was investigated by T. Kato ([1]). Under certain assumptions he showed, using the theory of semigroups, that the solution of the homogeneous equation

$$\frac{dx}{dt} + A(t)x = 0 \tag{2}$$

with initial condition x_0 from the domain of definition $D(A)$ of the operator $A(s)$:

$$x(s) = x_0 \tag{3}$$

can be represented in the form

$$x(t) = U(t,s)x_0, \tag{4}$$

where $U(t,s)$ is an operator, bounded and jointly continuous in t and s. The solution of equation (1) is then constructed by Kato in the form

$$x(t) = U(t,0)x_0 + Qf(t), \tag{5}$$

where

$$Qf(t) = \int_0^t U(t,s)f(s)\,ds \tag{6}$$

under stringent restrictions on the function $f(t)$.

In the authors' paper ([2]), the operator Q was subjected to further study. In particular, it was shown that (5) gives a solution of equation (1) for any continuously differentiable $f(t)$. Judging from the review in Math. Rev., **17**, No. 1 (1956), Kato also obtained an analogous result. With the aid of the operator Q, in ([2]) an equation of type (1) with nonlinearities was also considered.

In the particular case considered below, the operators U and Q can be studied in greater detail. We note that by other methods equation (1) with operators $A(t)$ acting in a Hilbert space was studied in the works of M. Vishik and O. Ladyzhenskaya ([3,4]).

1. We assume that the following conditions are satisfied.

a) The operators $A(t)$ are self-adjoint and uniformly bounded below:

$$(A(t)x, x) \geq (x, x).$$

b) The operators $A^{-\alpha}(t)$, for $0 \leq \alpha \leq 1$, are strongly differentiable with respect to t, and the operators

$$C_\alpha(t) = A^\alpha(t)\frac{d}{dt}A^{-\alpha}(t) \tag{7}$$

are bounded uniformly with respect to α and t.

c) The operator $C_1(t)$ is strongly continuous with respect to t and bounded.

Conditions b) and c) are satisfied in an obvious way if $A(t)$ does not depend on t. Condition b) is easily verified if the operators $A(t)$ commute with one another for different t.

It is of interest to ask under what conditions b) follows from a) and c).

A bounded operator F acting in H will be called absolutely bounded with respect to an orthonormal basis $\{e_i\}$ if the infinite matrix with entries $|(Fe_i, e_j)|$ defines a bounded operator in Hilbert space. We shall denote the norm of this operator by $N(F, e_i)$. We note that operators having finite absolute norm (see (5)) are absolutely bounded with respect to any basis.

Theorem 1. Suppose that the operator $A^{-1}(t)$ is strongly differentiable with respect to t and is completely continuous for each t. Suppose that the operators $C_1(t)$ are absolutely bounded with respect to the basis $\{e_i(t)\}$ formed from eigenvectors of the operators $A(t)$, and that the function $N[C_1(t), e_i(t)]$ is uniformly bounded on $[0, b]$.

Then b) follows from conditions a) and c).

The proof of the theorem is based on a generalization of the formula for differentiating, with respect to a parameter, functions of self-adjoint operators, obtained in (6).

2. Formula (4) makes it possible to construct solutions of problem (2)–(3) also in the case when $x_0 \in \overline{D(A)}$.

Theorem 2. The function $x(t)$ defined by formula (4) satisfies, for $t > s$, the homogeneous equation (2) for any $x_0 \in H$.

A more general assertion for the case of a constant operator A was established in (7) (see also (8)).

Theorem 2 is based on the fact that $U(t,s)x_0 \in D(A)$ for $t > s$ and for any $x_0 \in H$. In the proof of the subsequent theorems a stronger fact is used, which, as it seems to us, is of independent interest.

Theorem 3. For $t > s$ and $0 \leq \alpha < 2$, the operators $A^\alpha(t)U(t,s)$ are bounded, and the estimate

$$\left\|A^\alpha(t)U(t,s)\right\| \leq M(t-s)^{-\alpha}. \tag{8}$$

holds.

Estimate (8) is also valid for $\alpha = 2$, if $C(t)$ satisfies the Lipschitz condition $\operatorname{Lip}\beta$:

$$\|C(t) - C(s)\| \leq L|t-s|^\beta. \tag{9}$$

If the operator A is constant, then (8) holds for all $\alpha > 0$.

It follows from Theorem 3, in particular, that when condition (9) is satisfied, the solution of the homogeneous equation (2) is twice differentiable with respect to t for $t > s$.

3. **Theorem 4.** Suppose the function $f(t)$ satisfies the condition $\operatorname{Lip}\varepsilon$, $\varepsilon \leq 1$. Then formula (5) determines, for $t > 0$, a solution of equation (1) for any initial condition $x_0 \in H$.

If $x_0 \in D(A)$, then this solution has the property

$$\left\|A^\alpha(t)\frac{dx}{dt}\right\| \leq M|t|^{-\alpha}$$

for $\alpha < \varepsilon$.

In this theorem, for the case of equations in Hilbert space, the corresponding result from the paper (2), where equations in Banach spaces were considered, is strengthened.

4. Let C denote the space of functions $f(t)$, continuous on $[0, b]$, with values in H, with norm $\|f\|_C = \max \|f(t)\|$, and let C_0^1 denote the space of continuously differentiable functions, equal to zero for $t = 0$, with norm $\|f\|_{C_0^1} = \max \|f'(t)\|$. The operator Q, defined by formula (6), acts and is bounded in these spaces.

Theorem 5. If $f(t) \in C$, then

$$\|Qf(t + \Delta t) - Qf(t)\| \leq K_1 \Delta t \, |\ln \Delta t| \, \|f\|_C.$$

If $f(t) \in C_0^1$, then

$$\left\| \frac{d}{dt} Qf(t + \Delta t) - \frac{d}{dt} Qf(t) \right\| \leq K_2 \Delta t \, |\ln \Delta t| \, \|f\|_{C_0^1}.$$

From Theorems 4 and 5 there follows the theorem:

Theorem 6. If the operator $A^{-1}(t)$ is completely continuous, then the operator Q is completely continuous in the spaces C and C_0^1.

5. We turn to the consideration of the nonlinear equation

$$\frac{dx}{dt} + A(t)x = f(t, x). \tag{10}$$

We shall assume that the operator $A(t)$ satisfies conditions a), b), c), and that for each t the operator $A^{-1}(t)$ is completely continuous. Concerning $f(t, x)$, we shall assume that it is a continuous bounded operator with values in H, defined on $[0, b] \times T$, where T is the ball of radius r with center at x_0.

In parallel with equation (10), consider the integral equation

$$x(t) = U(t, 0)x_0 + Qf[t, x(t)]. \tag{11}$$

If the solution $x(t)$ of the integral equation (11) is such that $f[t, x(t)]$ satisfies the condition $\mathrm{Lip}\, \alpha$ $(\alpha \leq 1)$, then, by Theorem 4, the function $x(t)$ is a solution of the differential equation (10), satisfying the initial condition $x(0) = x_0$.

Theorem 7. Equation (11) has a continuous solution on some interval $(0, h]$ $(h \leq b)$. If the condition

$$\|f(t + \Delta t, x + \Delta x) - f(t, x)\| \leq K\left(|\Delta t|^\alpha + \|\Delta x\|^\alpha\right) \quad (\alpha \leq 1)$$

is fulfilled, then every continuous solution of equation (11) is, for $t > 0$, a solution of equation (10).

6. The requirement of continuity of the operator $f(t, x)$ is, for a Hilbert space, very stringent.

If, for example, one considers the operator $f(x) = x^2$, then on L_2 it is not everywhere defined and is not continuous. However, this operator is bounded and continuous in the space of continuous functions, which is a part of L_2. Therefore it is important to study the case of operators $f(t, x)$ that are bounded and continuous on some linear set M of the space H, which is a Banach space with respect to another norm $\|x\|_M$. In this case the inequality

$$\|x\| \leq K\|x\|_M \quad (x \in M) \tag{12}$$

usually holds.

Suppose that the operator $f(t, x)$ acts from M into H and is continuous, while the operator $U(t, 0)x_0 + Qy$ acts from H into M. Then in equation (10) one may make the substitution

$$x(t) = U(t, 0)x_0 + Qy(t) \tag{13}$$

and pass to the integral equation

$$y(t) = f[t, U(t, 0)x_0 + Qy(t)] \tag{14}$$

with a continuous operator in H on the right-hand side.

Theorem 8. Let the operators $A(t)$ satisfy the conditions of the preceding paragraph and, in addition, let the operators $A^{-\varepsilon}(t)$, for some ε, $0 < \varepsilon < 1$, act from H into M, with

$$\|A^{-\varepsilon}(t)x\|_M \leq q\|x\|_H \quad (x \in H),$$

where q does not depend on t. Let the operator $f(t, x)$, with values in H, be defined and continuous in a neighborhood of the point $\{0, x_0\}$ of the space $[0, b] \times M$. Finally, let $x_0 \in D(A^\varepsilon)$.

Then on some interval $[0, h]$, $h \leq b$, there exists a solution, continuous in H, of the integral equation (14). This solution, by formula (13), determines a solution $x(t)$ of equation (10) that is continuous in M and differentiable in H, provided the function $f(t, x)$ satisfies the condition

$$\|f(t + \Delta t, x + \Delta x) - f(t, x)\|_H \leq K\left(|\Delta t|^\alpha + \|\Delta x\|_M^\alpha\right).$$

7. In conclusion we give an example. Consider the mixed problem with zero boundary conditions for the heat equation with nonlinearity:

$$\frac{\partial u}{\partial t} = \Delta u + f(t, s, u)$$

in a bounded domain of n-dimensional space. For simplicity we shall assume that the function $f(t, s, u)$ is continuously differentiable with respect to all $n+2$ variables. From Theorem 8 there follows a local theorem on the existence of bounded solutions of this mixed problem for spaces of dimension $n = 2, 3$. If the dimension $n > 3$, then the existence of solutions can be proved under restrictions on the growth of the function $f(t, s, u)$:

$$|f(t, s, u)| \leq a + b|u|^m,$$

where $m < 1 + 4/(n - 4)$.

In the investigation of the mixed problem just presented, the results of V. A. Il' in [9] and M. Riesz on fractional powers of the Laplace operator and the general theorems on the splitting of a linear operator [10] are used essentially.

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REFERENCES

[1] T. Kato, J. Math. Soc. Japan, **5**, No. 2 (1953).
[2] M. A. Krasnosel' skii, S. G. Krein, P. E. Sobolevskii, DAN, **111**, No. 1 (1956).
[3] M. I. Vishik, Matem. sborn., **139**, No. 1 (1956).
[4] O. A. Ladyzhenskaya, DAN, **102**, No. 2 (1955).
[5] V. I. Smirnov, *A Course of Higher Mathematics*, 5, 1947.
[6] Yu. L. Daletskii, S. G. Krein, *Proceedings of a Seminar on Functional Analysis*, Voronezh, vol. 1, 1956.
[7] P. D. Lax, A. N. Milgram, Ann. Mathem. Stud., No. 33, 167 (1953).
[8] E. Hille, *Functional Analysis and Semigroups*, IL, 1951.
[9] V. A. Il' in, DAN, **105**, No. 1 (1955).
[10] M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, 1956.

Note: Figure translations are in progress. See original paper for figures.