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Abstract

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MATHEMATICS

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A PRIORI ESTIMATES AND REGULARITY OF GENERALIZED SOLUTIONS OF THE MONGE-AMPÈRE EQUATIONS

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1. In the present paper we consider a priori estimates for solutions of the equations

$$rt - s^2 = \varphi(x, y, z, p, q), \tag{1}$$

where $\varphi(x, y, z, p, q) \geqslant k_0 = \text{const} > 0$ is a function continuously differentiable m times ($m \geqslant 3$). These estimates are a generalization of estimates obtained by S. N. Bernstein (^{1}) for the equations $rt - s^2 = C$ ($C = \text{const} > 0$) and by A. V. Pogorelov for the equations $rt - s^2 = \varphi(x, y)$ ($\varphi(x, y) \geqslant \text{const} > 0$). The a priori estimates obtained make it possible to establish the regularity of generalized solutions of equations (1).

2. A priori estimates of the modulus of the solution and of its first derivatives. Consider in the disk $\Omega : x^2 + y^2 \leqslant R^2$ with boundary Γ an analytic solution $z(x, y)$ of equation (1), which on Γ turns into an analytic function $\psi(\theta)$ of the polar angle θ. The function $\varphi(x, y, z, p, q)$ is assumed analytic in Ω with respect to x, y and for all finite z, p, q. Further, we assume that $\varphi_z'(x, y, z, p, q) \geqslant 0$ for all functions $z(x, y)$ continuously differentiable in $\Omega + \Gamma$*. Introduce the function

$$N_\varphi(\nu_1, \nu_2) = \sup \varphi(x, y, z, p, q),$$

where the exact upper bound is taken in the domain $x^2 + y^2 \leqslant R$, $|z| \leqslant \nu_1$, $\operatorname{grad}^2 z \leqslant \nu_2$.

Let γ be the edge of the surface $z = z(x, y)$ and P an arbitrary point of γ. Denote by $\mu(P) = \inf(a^2 + b^2)$, where the exact lower bound is taken over all planes $z = ax + by + c$ passing through the point P and tangent to γ at this point and leaving the curve γ below them. Let

$$M(\psi) = \sup_{P \in \gamma} \mu(P).$$

An estimate for $M(\psi)$ was obtained by S. M. Bernstein (^{1}) in terms of $\psi(\theta)$ and its derivatives up to the third order. Below an estimate for $M(\psi)$ will be given from other considerations.

Theorem 1. *Let the conditions formulated above be fulfilled with respect to the solution $z(x, y)$ and the function $\varphi(x, y, z, p, q)$. Then, if*

$$4R^2 N_\varphi(\nu_1(R), \nu_2(R)) \leqslant 1, \tag{2}$$

where

$$\nu_1(R) = (2-\sqrt{3})R + R^2 + 3\max_{0\leqslant\theta\leqslant 2\pi}|\psi(\theta)|, \quad \nu_2(R) = \frac{2}{3} + 4R^2 + 4M(\psi),$$

then the inequalities hold

$$\max_{\Omega+\Gamma}|z| \leqslant (2-\sqrt{3})R + R^2 + 3\max_{0\leqslant\theta\leqslant 2\pi}|\psi(\theta)|,$$

$$\max_{\Omega+\Gamma}(\operatorname{grad}^2 z) \leqslant \frac{2}{3} + 4R^2 + 4M(\psi).$$

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* The condition $\varphi(x, y, z, p, q) \geqslant k_0 = \text{const} > 0$, introduced in item 1, is assumed to be fulfilled.

Theorem 1 is valid if the condition $\varphi'_z(x, y, z, p, q) \geqslant 0$ is replaced by the condition $\varphi(x, y, z, p, q) \leqslant \Phi_0 \widetilde{R}(p, q)$, where $\Phi_0 = \text{const} > 0$ and $\widetilde{R}(p, q)$ is a continuously differentiable positive function, and condition (2) is replaced by the condition

$$4R^2\Phi_0\widetilde{N}_R(\nu_2(R)) \leqslant 1/\Phi_0,$$

where

$$\widetilde{N}_R(\nu_2) = \max_{p^2+q^2\leqslant\nu_2}\widetilde{R}(p, q).$$

3. A priori estimates of the second derivatives of solutions in a closed disk. Let, in equation (1), the right-hand side satisfy the inequality

$$\varphi(x, y, z, p, q) \geqslant k_0 = \text{const} > 0$$

for all continuously differentiable functions $z(x, y)$ in $\Omega+\Gamma$, and let this function still be analytic in $(x, y) \in \Omega$ and for all finite z, p, q; the fulfillment of the inequality $\varphi_z(x, y, z, p, q) \geqslant 0$ is not required in the present section. Let, further, $\psi(\theta)$ be an analytic function and let $z(x, y)$ be an analytic solution of equation (1) satisfying the condition $z(x, y)|_\Gamma = \psi(\theta)$. Suppose that we know the estimates

$$\max_{\Omega+\Gamma}|z| \leqslant L_0, \qquad \max_{\Omega+\Gamma}(\operatorname{grad}^2 z) \leqslant L_1.$$

We set the problem of obtaining analogous estimates for the second derivatives of the function $z(x, y)$. The estimates of the second derivatives in the closed domain are carried out separately on the boundary and inside the disk. Let ρ, θ be polar coordinates in the x, y-plane. The estimate of $z_{\theta\theta}$ on Γ is trivial. The estimates for $z_{\rho\theta}$ and $z_{\rho\rho}$ on Γ depend on the following quantities: a) the maximum modulus of $\psi(\theta)$ and its derivatives up to and including the fourth order; b) the numbers L_0, L_1, and k_0; c) the number

$$\Phi_0 = \max\{\sup|\varphi_\theta|, \sup|\varphi_z|, \sup|\varphi_{zp}|, \sup|\varphi_{z\theta}|\},$$

where the exact upper bound of each of the functions under consideration is taken in the domain

$$R/2 \leqslant \rho \leqslant R, \quad 0 \leqslant \theta \leqslant 2\pi, \quad |z| \leqslant L_0, \quad \operatorname{grad}^2 z \leqslant L_1.$$

The estimates of the second derivatives inside Ω are obtained by the method of auxiliary functions introduced by S. N. Bernstein. For our purposes a construction introduced by A. V. Pogorelov in (2) is convenient; namely, the auxiliary functions for obtaining estimates of $r^2 + s^2$ and $s^2 + t^2$ have the form

$$w_1 = \lambda_1 r, \qquad w_2 = \lambda_2 t,$$

where λ_1 and $\lambda_2 > 0$ are certain known functions of x, y. In our case we take

$$\lambda_1 = \lambda_2 = x^2 + y^2 + R^2,$$

where R is the radius of the disk Ω. If the radius of the disk Ω is sufficiently small, then estimates for $|r|, |s|, |t|$ can be obtained in terms of $\varphi(x, y, z, p, q)$ and its derivatives up to and including the second order in the domain

$$x^2 + y^2 \leqslant R^2, \quad |z| \leqslant L_0, \quad \operatorname{grad}^2 z \leqslant L_1.$$

The final results concerning estimates of the second derivatives may be formulated as follows.

Theorem 2. *Let, with respect to the solution $z(x, y)$ and the function $\varphi(x, y, z, p, q)$, the conditions formulated in Section 3 be fulfilled. Then, if the radius R of the disk Ω satisfies the inequality $4R^2 A_0 \leqslant 1$, where*

$$A_0 = \max\{\sup|\varphi_{pp}|, \sup|\varphi_{pq}|, \sup|\varphi_{qq}|\},$$

the exact upper bounds of the indicated functions being taken in the domain $(x, y) \in \Omega$, $|z| \leqslant L_0$, $\operatorname{grad}^2 z \leqslant L_1$, then for $|r|, |s|, |t|$ in $\Omega + \Gamma$ one can obtain estimates depending only on $\psi(\theta)$ and its derivatives up to the fourth order, and on the upper bounds of the moduli of $\varphi(x, y, z, p, q)$ and its partial derivatives with respect to all variables of the first two orders in the domain

$$x^2 + y^2 \leqslant R^2, \quad |z| \leqslant L_0, \quad \operatorname{grad}^2 z \leqslant L_1.$$

Denote by

$$T_\varphi(\nu_1, \nu_2) = \max\{\sup|\varphi_{pp}|, \sup|\varphi_{pq}|, \sup|\varphi_{qq}|\},$$

where the exact upper bounds of $|\varphi_{pp}|, |\varphi_{pq}|, |\varphi_{qq}|$ are taken in the domain

$$(x, y) \in \Omega, \quad |z| \leqslant \nu_1, \quad \operatorname{grad}^2 z \leqslant \nu_2.$$

From the results of S. N. Bernstein on the solvability of the Dirichlet problem for equations of elliptic type (1), and from Theorems 1 and 2, the following theorem follows.

Theorem 3. *Consider in the disk $\Omega : x^2 + y^2 \leqslant R^2$ with boundary Γ the equation*

$$rt - s^2 = \varphi(x, y, z, p, q).$$

Let the function $\varphi(x, y, z, p, q)$ satisfy the following conditions: a) $\varphi(x, y, z, p, q)$ is an analytic function of $(x, y) \in \Omega$ and for all finite z, p, q; b) $\varphi_z(x, y, z, p, q) \geqslant 0$

and $\varphi(x, y, z, p, q) \geq k_0 = \text{const} > 0$ on all functions $z(x, y)$ continuously differentiable in $\Omega + \Gamma$. Let $\psi(\theta)$ be an analytic function of the polar angle θ, prescribed on Γ. Then, if the inequalities

$$4R^2 N_\varphi(\nu_1(R), \nu_2(R)) \leq 1, \quad 4R^2 T_\varphi(\nu_1(R), \nu_2(R)) \leq 1,$$

are satisfied, where

$$\nu_1(R) = (2-\sqrt{3})R + R^2 + \frac{3}{0 \leq \theta \leq 2\pi} \max|\psi(\theta)|, \quad \nu_2(R) = \frac{2}{3} + 4R^2 + 4M(\psi),$$

then there exists an analytic solution $z(x, y)$ of the equation

$$rt - s^2 = \varphi(x, y, z, p, q),$$

turned with its convexity toward $z < 0$ and reducing on Γ to the function $\psi(\theta)$.

Theorem 3 remains valid if the condition $\varphi_z \geq 0$ is replaced by the condition

$$\varphi(x, y, z, p, q) \leq \Phi_0 \widetilde{R}(p, q), \tag{3}$$

and the condition $4R^2 N_\varphi(\nu_1(R), \nu_2(R)) \leq 1$ by the condition $4R^2 N_{\widetilde{R}}(\nu_2(R)) \leq 1/\Phi_0$.

We note that Leray (3) obtained a priori estimates for solutions of the equations

$$rt - s^2 + g(r, s, t, p, q, x, y, z) + h(r, s, t, p, q, x, y, z) = 0,$$

where $g(r, s, t, p, q, x, y, z)$ is a homogeneous function of the first degree with respect to r, s, t, and $h(r, s, t, p, q, z, x, y)$ and its derivatives are estimated by bounded functions of p, q, z, x, y. Namely, in (3) estimates were obtained for the first derivatives in the interior and for the second derivatives in the closed domain D, in terms of the boundary conditions, the properties of the function $g(r, s, t, p, q, z, x, y)$ and its derivatives, and also in terms of $\max|z|$ in the domain D and $\max(p^2 + q^2)$ on the boundary D. The estimates established in § 3 are obtained under conditions different from those in Leray' s work (3).

4. Let $z(x, y)$ be an m-times continuously differentiable solution $(m \geq 6)$ of equation (1), turned with its convexity toward $z < 0$, and let, in the disk $\Omega : x^2 + y^2 \leq R^2$, where $z(x, y)$ is prescribed, the estimates

$$\max_{\Omega+\Gamma} |z| \leq L_0, \quad \max_{\Omega+\Gamma} (\operatorname{grad}^2 z) \leq L_1$$

be known. The function $\varphi(x, y, z, p, q)$ is three times continuously differentiable with respect to $(x, y) \in \Omega$ and for all finite z, p, q. Further, $\varphi(x, y, z, p, q) \geq k_0 = \text{const} > 0$ on all functions $u(x, y)$ continuously differentiable in $\Omega + \Gamma$.

Theorem 4. Let the conditions formulated in § 4 be satisfied with respect to the function $\varphi(x, y, z, p, q)$ and the solution $z(x, y)$ of equation (1). Then, if the radius R of the disk Ω satisfies the inequality $4R^2 T_\varphi(L_0, L_1) \leq 1$, estimates can be given for the second, third, and fourth derivatives of the function $z(x, y)$ in the disk

$$G_\delta : \quad x^2 + y^2 \leq (R - \delta)^2$$

depending only on the following quantities: $\delta > 0$, k_0, L_0, L_1, the upper bound of the moduli of the function $\varphi(x, y, z, p, q)$ and of its derivatives up to the third order inclusive in the domain

$$x^2 + y^2 \leq (R - \delta)^2, \quad |z| \leq L_0, \quad \operatorname{grad}^2 z \leq L_1.$$

Theorem 5. Let, in the disk $\Omega : x^2 + y^2 \leq R^2$, the equation

$$rt - s^2 = \varphi(x, y, z, p, q)$$

be given, where the function $\varphi(x, y, z, p, q)$ is three times continuously differentiable with respect to $(x, y) \in \Omega$ and for all finite values z, p, q; further, $\varphi(x, y, z, p, q) \geq k_0 = \text{const} > 0$, $\varphi_z(x, y, z, p, q) \geq 0$ on all functions $u(x, y)$ continuously differentiable in $\Omega + \Gamma$. Let $\psi(\theta)$ be a continuously differentiable function of the polar angle θ, which admits a uniform approximation by analytic functions $\psi_n(\theta)$ together with the first derivatives and, moreover, such that the numbers $M(\psi_n)$ are uniformly bounded above by the number M_0. Let the inequalities* be satisfied:

$$5R^2 N_\varphi(\nu_1(R), \nu_2(R)) \leq 1, \quad 5R^2 T_\varphi(\nu_1(R), \nu_2(R)) \leq 1. \tag{4}$$

Then there exists a function $z(x, y)$, three times continuously differentiable inside Ω, which satisfies inside Ω the equation

$$rt - s^2 = \varphi(x, y, z, p, q),$$

reduces on Γ to $\psi(\theta)$. The function $z(x, y)$ is turned with its convexity toward $z < 0$.

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* The quantities $\nu_1(R), \nu_2(R)$ are defined in § 1; in the quantity $\nu_2(R)$ the number M_0 is substituted here in place of $M(\psi)$.

As above, the condition $\varphi_z \geq 0$ can be replaced by condition (3). Theorem 5 will then be valid if (4) is replaced by the inequality $5R^2 N_{\widetilde{R}}(\gamma_2(R)) \ll 1/\Phi_0$.

5. As is known, every convex function has a second differential almost everywhere. We shall now understand by a generalized solution of equation (1), where $\varphi(x, y, z, p, q) \geq k_0 = \text{const} > 0$ is a continuous function of its variables, a convex function $z(x, y)$ satisfying equation (1) almost everywhere. Let D be a convex domain in which a generalized solution of equation (1) is given; then for every closed subdomain D_δ, separated from the boundary of D by a positive distance $\delta > 0$, the function $z(x, y)$ is continuously differentiable and $\sup_{D_\delta} \operatorname{grad}^2 z < +\infty$. Let X be an interior point of D_δ, and let U be a circular neighborhood of the point X contained in D_δ. Then the continuously differentiable function $\psi(\theta) = z|_{\operatorname{bd} U}$ admits a uniform approximation by analytic functions $\psi_n(\theta)$, together with their first derivatives, and in such a way that $M(\psi_n) \leq \sup_{D_\delta}(1 + \operatorname{grad}^2 z)$.

Let the function $\varphi(x, y, z, p, q)$ satisfy the condition $\varphi'_z \geq a = \text{const} > 0$ on all continuously differentiable functions $u(x, y)$, or the condition $\varphi(x, y, z, p, q) \leq \Phi_0 \widetilde{K}(p, q)$, where $\Phi_0 = \text{const} > 0$, $\widetilde{R}(p, q) \geq R_0 = \text{const} > 0$. Then one can prove that if the generalized solutions $z_1(x, y)$ and $z_2(x, y)$ of equation (1), with convexity directed toward $z < 0$, coincide on the boundary of a convex domain D, then $z_1(x, y)$ and $z_2(x, y)$ coincide throughout the whole domain D (the functions $z_1(x, y)$ and $z_2(x, y)$ are considered to be defined and continuous in the closed domain $\overline{D}$).

From what has been set forth above, the following theorem follows.

Theorem 6. *Let the function $z(x, y)$ be a generalized solution of equation (1) in a convex domain D, and let the function $\varphi(x, y, z, p, q)$ satisfy the conditions formulated in § 5. Then, if the function $\varphi(x, y, z, p, q) \geq k_0 > 0$ is three times continuously differentiable with respect to x, y in the domain Ω and for all finite z, p, q, then the function $z(x, y)$ is at least four times continuously differentiable inside D.*

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