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Abstract

Full Text

MATHEMATICS

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ON THE MAXIMUM PRINCIPLE FOR SOLUTIONS OF STRONGLY ELLIPTIC SYSTEMS OF SECOND ORDER

(Presented by Academician S. L. Sobolev on 9 IV 1957)

The maximum principle plays a major role in the study of solutions of elliptic equations of second order. Solutions of elliptic equations of higher orders, as well as solutions of elliptic systems, as is known, do not satisfy the usual maximum principle. However, as will be shown below, solutions of strongly elliptic systems of second order satisfy a certain integral maximum principle.

Thus, let the k -dimensional vector $u(x, t)$ satisfy the strongly elliptic system

$$\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^n A_{0i} \frac{\partial^2 u}{\partial x_i \partial t} + \sum_{i,j=1}^n A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0. \tag{1}$$

The vector u is defined in the cylinder $x \in \Omega$, $0 \leq t \leq l$ and vanishes on the boundary of Ω ; A_{ij} are constant symmetric matrices; x is an n -dimensional vector.

We shall show that the integral $\int_{\Omega} u^2(x, t) dx$ attains its maximum only at the boundary of the interval $[0, l]$.

Let $\int_{\Omega} u^2(x, t) dx = f(t)$. Differentiating the function $f(t)$, we obtain

$$f''(t) = 2 \int_{\Omega} u_t^2 dx + 2 \int_{\Omega} u u_{tt} dx. \tag{2}$$

In view of (1),

$$\int_{\Omega} u u_{tt} dx = - \int_{\Omega} \left(u, \sum_{i=1}^n A_{0i} \frac{\partial^2 u}{\partial x_i \partial t} + \sum_{i,j=1}^n A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \right) dx =$$

$$= \int_{\Omega} \left[\sum_{i=1}^n \left(A_{0i} \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i} \right) + \sum_{i,j=1}^n \left(A_{ij} \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right) \right] dx,$$

and hence

$$f''(t) = 2 \int_{\Omega} \left[u_t^2 + \sum_{i=1}^n (A_{0i} u_t, u_{x_i}) + \sum_{i,j=1}^n (A_{ij} u_{x_i}, u_{x_j}) \right] dx. \quad (3)$$

From the definition of strong ellipticity it follows that the integrand on the right-hand side of equality (3) is nonnegative, and therefore

$$f''(t) \geq 0. \quad (4)$$

It follows from inequality (4) that on the interval $[0, l]$ the function $f(t)$ satisfies the inequality

$$f(t) \leq f(0) + f(l)t,$$

from which we obtain the stated maximum principle.

In the case of a strongly elliptic second-order system of more general form and with variable coefficients, a generalized maximum principle holds. Namely, the following theorem holds:

Theorem. Let the k -dimensional vector $u(x, t)$ satisfy the system

$$\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^n A_{i0} \frac{\partial^2 u}{\partial t \partial x_i} + \sum_{i,j=1}^n A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + B_0 \frac{\partial u}{\partial t} + \sum_{i=1}^n B_i \frac{\partial u}{\partial x_i} + Cu = 0 \quad (5)$$

in the cylinder $x \in \Omega$, $0 \leq t \leq l$, and vanish on the boundary of Ω .

With respect to the matrices A, B, C we assume that:

- 1) the matrices A are symmetric and satisfy the Lipschitz condition, so that throughout the closed cylinder the inequality holds

$$|A(x', t') - A(x'', t'')| \leq \lambda_1 [|x' - x''| + |t' - t''|];$$

- 2) the matrices B, C are summable on all k -dimensional manifolds and bounded, so that throughout the closed cylinder

$$\|B\| \leq \lambda_2, \quad \|C\| \leq \lambda_3;$$

3) system (5) is strongly elliptic, i.e. for any real k -dimensional vectors $\xi, \eta_1, \dots, \eta_n$ throughout the closed cylinder

$$\xi^2 + \sum_{i=1}^n (A_{i0}\xi, \eta_i) + \sum_{i,j=1}^n (A_{ij}\eta_i, \eta_j) \geq \alpha \left[|\xi|^2 + \sum_{i=1}^n |\eta_i|^2 \right]$$

(λ and α are constants).

Then there exists an h , depending only on the constants λ, α and the numbers k, n , such that for any t_0 , $0 \leq t_0 \leq l - h$, and any t , $t_0 \leq t \leq t_0 + h$, the inequality holds

$$\int_{\Omega} u^2(x, t) dx \leq 2 \max \left[\int_{\Omega} u^2(x, t_0) dx, \int_{\Omega} u^2(x, t_0 + h) dx \right].$$

From the stated theorem there immediately follows the uniqueness of the solution of the Dirichlet problem for system (5) in sufficiently small domains.

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Note: Figure translations are in progress. See original paper for figures.

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