A MIXED PROBLEM FOR THE EQUATION $\psi_{\sigma\sigma}-K(\sigma)\psi_{\theta\theta}=0$ WITH CAUCHY DATA ON THE CURVE $\theta=s(\sigma)$

1957

SovietRxiv

Abstract

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MATHEMATICS

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A MIXED PROBLEM FOR THE EQUATION $\psi_{\sigma\sigma} - K(\sigma)\psi_{\theta\theta} = 0$ WITH CAUCHY DATA ON THE CURVE $\theta = s(\sigma)$

(Presented by Academician V. I. Smirnov on 28 XII 1956)

1. In the note [1] we considered the boundary-value problem (I_0) for the equation

$$\psi_{\sigma\sigma} - K(\sigma)\psi_{\theta\theta} = 0, \qquad K(\sigma) \in C^{(2)}, \tag{1_0}$$

in the hyperbolic strip $(K(\sigma) > 0)$

$$S\{\sigma_0 \le \sigma \le \sigma_1,\ -\infty < \theta < +\infty\}$$

with data $\psi = \text{const}$ on the straight lines $\sigma = \sigma_0$, $\sigma = \sigma_1$ and $\psi = \psi(\sigma)$ on a segment of a characteristic between these straight lines. We now investigate the problem obtained from (1_0) if the last condition is replaced by Cauchy data on a segment of some noncharacteristic curve $\theta = s(\sigma)$. First we subtract the constant values of ψ on $\sigma = \sigma_0$, $\sigma = \sigma_1$ by means of a particular solution ψ_0 of equation (1_0) [1] and transform equation (1_0) to the form

$$L(v) \equiv v_{\zeta\zeta} - v_{\vartheta\vartheta} + N(\zeta)v = 0,$$

where

$$v(\zeta, \vartheta) = [\psi(\sigma, \theta) - \psi_0]K^{1/4}(\sigma); \qquad N(\zeta) = -K^{-1/4}(\sigma)\frac{d^2K^{1/4}(\sigma)}{d\zeta^2};$$

$$c\,d\zeta = \sqrt{K(\sigma)}\,d\sigma, \qquad c\,d\vartheta = d\theta, \qquad c = \int_{\sigma_0}^{\sigma_1} \sqrt{K(\sigma)}\,d\sigma.$$

Then the formulation of the problem under consideration will have the following form:

$$L(v) = 0; \qquad v|_{\zeta=0} = v|_{\zeta=1} = 0; \qquad v|_{\vartheta=l(\zeta)} = p(\zeta); \qquad v_\vartheta|_{\vartheta=l(\zeta)} = q(\zeta); \tag{C}$$

the functions $l(\zeta)$, $p(\zeta)$, $q(\zeta)$ are defined on $[0,1]$; $l(\zeta)$ satisfies the condition

$$|\, l(\zeta') - l(\zeta'')\,| < |\zeta' - \zeta''|; \qquad \zeta', \zeta'' \in [0,1]; \qquad \zeta' \neq \zeta''. \tag{2}$$

In the present note we shall set forth a method of effective solution of problem (C) under various restrictions on the functions N, l, p, q.

2. Following [1], we shall seek the solution of problem (C) in the form of a series of particular solutions of equation (1)

$$v = \sum_{n=-\infty}^{\infty}{}' c_n B_n(\zeta) \exp(-i\lambda_n \vartheta), \tag{3}$$

in which, for $n > 0$, λ_n and $B_n(\zeta)$ are the eigenfunctions of the Sturm-Liouville problem

$$B_n'' + [\lambda_n^2 + N(\zeta)]B_n = 0, \tag{4}$$

$$B_n(0) = B_n(1) = 0, \tag{5}$$

where the λ_n are numbered in increasing order,

$$\lambda_{-n} = -\lambda_n, \qquad B_{-n}(\zeta) = -B_n(\zeta). \tag{6}$$

In ([1]) it is shown that 0 is not an eigenvalue.

If the series (3) converges uniformly in S, then the conditions on the straight lines $\zeta = 0$, $\zeta = 1$ are satisfied by virtue of (5). If, in addition, the series (3) can be differentiated once with respect to ϑ in a neighborhood of l, then satisfaction of the initial conditions on l leads to expansions of two arbitrary functions $p(\zeta)$ and $q(\zeta)$ in the series

$$p(\zeta) \approx \sum_{n=-\infty}^{\infty}{}' c_n \bar{z}_n, \tag{7}$$

$$q(\zeta) \approx \sum_{n=-\infty}^{\infty}{}' c_n(-i\lambda_n)\bar{z}_n, \tag{8}$$

where

$$z_n = B_n(\zeta) \exp[i\lambda_n l(\zeta)], \tag{9}$$

and $\bar{z}_n$ denotes the conjugate expression.

If $l(\zeta)$ has a second derivative, then the function $\bar{z}_n(\zeta)$ satisfies the equation ([1], equation (4))

$$\bar{z}_n'' + 2i\lambda_n l' \bar{z}_n' + \bar{z}_n\{N + i\lambda_n l'' + \lambda_n^2(1 - l'^2)\} = 0. \tag{10}$$

From (10) and the corresponding equation for z_m it is easy to derive the orthogonality relation

$$\int_0^1 \{2l' z_m' \bar{z}_n + z_m \bar{z}_n [l'' + i(1 - l'^2)(\lambda_m + \lambda_n)]\} \, d\zeta = 0. \tag{11}$$

With the aid of (11), when $p'(\zeta)$ and $q(\zeta)$ are integrable, one can formally determine the coefficients c_n in (7) and (8), which, after integration by parts of the term containing $l''(\zeta)$, are given by the formula

$$c_n = \frac{i}{2\lambda_n} \int_0^1 e^{i\lambda_n l} \{B_n [p'l' + q(1 - l'^2) - i\lambda_n p] - B_n' p l'\} \, dy. \tag{12}$$

To justify the operations performed, it remains to prove that the series (7) and (8) with coefficients c_n defined by formula (12) converge respectively to $p(\zeta)$ and $q(\zeta)$.

3. The expansion of a single function in a series in solutions of an equation of type (10) under certain boundary conditions was studied by J. D. Tamarkin ([2]). The first theorems on the simultaneous expansion of two arbitrary functions in eigenfunctions of a non-self-adjoint equation of a somewhat more general form than (10), under broad boundary conditions, are found in Langer' s work ([3]). M. V. Keldysh ([4]) generalized Langer' s results to operator equations of order n and introduced the concept of n-fold completeness of a system of eigen- and adjoint elements.

For a sufficiently smooth function $l(\zeta)$, one can use the results of Langer and Keldysh. However, the rather concrete form of our problem makes it possible to obtain analogous theorems under weaker restrictions on $l(\zeta)$. In particular, we shall not require even the existence of $l''(\zeta)$, which is necessary for writing equation (10), and in consequence of this we shall speak of expansion in the functions $\bar{z}_n$ defined by formulas (9), (4), and (5).

Consider contour integrals in the complex λ-plane

$$I_n^{(p)} = \frac{1}{2\pi} \oint e^{-i\lambda l(\zeta)} \left\{ \left(\frac{\chi(\zeta, \lambda)}{\omega(\lambda)} \right)_0^\zeta e^{i\lambda l(y)} [\varphi(y, \lambda)(p'l' + q(1 - l'^2) - i\lambda p) - \right.$$

$$-\varphi'(y,\lambda)pl'\right]dy+\frac{\varphi(\zeta,\lambda)}{\omega(\lambda)}\int_\zeta^1 e^{i\lambda l(y)}[\chi(y,\lambda)(p'l'+q(1-l'^2)-i\lambda p)-$$

$$-\chi'(y,\lambda)pl'\right]dy\Big\}\;d\lambda.$$

and $I_n^{(q)}$, which is obtained from $I_n^{(p)}$ by adding the factor $(-i\lambda)$ to the integrand. Here $\varphi(\zeta,\lambda)$ and $\chi(\zeta,\lambda)$ are solutions of equation (4) (for arbitrary λ) satisfying the conditions

$$\varphi(0,\lambda)=0,\quad \varphi'(0,\lambda)=1,\quad \chi(1,\lambda)=0,\quad \chi'(1,\lambda)=1;$$

$\omega(\lambda)=\varphi\chi'-\chi\varphi'$. (Detailed information about φ,χ,ω may be found in [5], Ch. 1, § 6.)

The integration is carried out over a sufficiently large circle with center at the origin and radius R_n, satisfying the inequality

$$\lambda_n<R_n<\lambda_{n+1}.$$

These integrals can be formally obtained by means of generalized Fourier transforms, analogous to F_+ and F_- from [5] (Ch. 1, § 3), with limits of integration with respect to ϑ respectively $(l(\zeta),+\infty)$ and $(-\infty,l(\zeta))$.

On the one hand, computing $I_n^{(p)}$ and $I_n^{(q)}$ as sums of residues, it is not hard to show that they are precisely partial sums of the series (7) and (8). Consequently, as $n\to\infty$ the integrals under consideration are formally identical with the series (7) and (8).

On the other hand, these integrals as $R_n\to\infty$ can be evaluated directly with the aid of asymptotic formulas for $\varphi,\chi,\varphi',\chi',\omega$, using the same techniques that Mishoe [6] applied in the case $l(\zeta)\equiv\zeta$.

Thus the following theorem is proved.

Expansion theorem. *If the functions $l'(\zeta),p'(\zeta),q(\zeta)$ have bounded variation on $[0,1]$, $p(0)=p(1)=0$, and $l(\zeta)$ satisfies condition (2), then the series (7) and (8), with coefficients determined by formula (12), converge respectively to $p(\zeta)$ and*

$$\frac{1}{2}[q(\zeta-0)+q(\zeta+0)].$$

In the course of the proof, two further results are obtained, valid when condition (2) is fulfilled and $l'(\zeta),p(\zeta)$ have bounded variation on $[0,1]$:

$$\sum_{n=-\infty}^{\infty}{}'\frac{-i}{2\lambda_n}\overline{z_n(\zeta)}\int_0^1 e^{i\lambda_n l(y)}p(y)\{i\lambda_n B_n(y)+l'(y)B_n'(y)\}\,dy=\frac{p(\zeta-0)+p(\zeta+0)}{2};$$

$$\sum_{n=-\infty}^{\infty}{}' \frac{1}{\lambda_n} z_n(\zeta) \int_0^1 z_n(y)p(y)\,dy = 0.$$

4. It remains to investigate the convergence of the series (3) and of its derivatives. Since the c_n are determined, this problem presents no difficulties. The structure of the convergence is easily clarified with the aid of the asymptotic formulas for λ_n and $B_n(\zeta)$ (of the type (12) and (13) from [1]) whenever specific properties of continuity and differentiability of the functions $N(\zeta), l(\zeta), p(\zeta), q(\zeta)$ are prescribed.

We list some general results.

Under the hypotheses of the expansion theorem, $c_n = O(n^{-2})$, and the series (3) converges uniformly.

If on $[0,1]$ there exist $N'', l'', q'', (ql' - p')''$ with bounded variation, and at the points $(0,0), (1, l(1))$ the compatibility conditions for the first and second derivatives are satisfied, then $c_n = O(n^{-4})$, and the series (3) gives a classical solution of problem (C) throughout the strip S.

In cases where the compatibility conditions are not satisfied, or $l, q, ql' - p'$ and their first derivatives have discrete discontinuities, but in each partial interval $l'', q'', (ql' - p')''$ still have bounded variation, one can improve the convergence of series (3) by the method of A. N. Krylov, separating out and summing in finite form the weakly convergent parts. Then there remains a series uniformly convergent after two term-by-term differentiations, and the entire pattern of discontinuities will be contained in the separated finite terms. The solution of problem (C) will be written in the form "elementary function $+\sum_n O(n^{-4})$" in each partial domain into which the strip S is divided by the characteristics carrying the above-mentioned discontinuities. In the whole strip S, the function (3) in such cases may be regarded as, in a certain sense, a generalized solution of problem (C). The concept of a generalized solution must also be used when the indicated improvement of convergence cannot be carried out.

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Received
25 XII 1956

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Note: Figure translations are in progress. See original paper for figures.