ON A CLASS OF OPERATORS NONLINEARLY DEPENDING ON A PARAMETER

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Abstract

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MATHEMATICS

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ON A CLASS OF OPERATORS NONLINEARLY DEPENDING ON A PARAMETER

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1. In the notes ($^{1-3}$), the spectral properties of linear operators depending polynomially on a parameter were investigated. In the present note we shall consider a class of operators depending nonlinearly on a parameter and possessing simple real poles.

Let X be a Hilbert space in which finite-dimensional self-adjoint operators of the form

$$H_i x = \sum_{k=1}^{\sigma_i} \frac{(x, \varphi_k^{(i)})\varphi_k^{(i)}}{\chi_k^{(i)}}, \qquad i = 1, 2, \dots,$$

are defined, where $\{\varphi_k^{(i)}\}$ is an orthonormal system of elements of X (for each i); $\chi_k^{(i)}$ are certain real numbers; the absolute norms and traces (^{4}) of the operators H_i, denoted respectively by $N(H_i)$ and $S_p(H_i)$, are assumed uniformly bounded: $N(H_i) \leqslant B,\ |S_p(H_i)| \leqslant M\ (B > 0,\ M > 0)$, $i = 1, 2, ...$; $\{a_i\}$ is a sequence of real numbers $a_i \neq 0$,

$$\sum_{i=1}^{\infty} \frac{1}{|a_i|} < +\infty, \qquad \alpha_i \chi_k^{(i)} < 0 \quad (i = 1, 2, \dots;\ k = 1, \dots, \sigma_i)$$

(from the latter it follows that $\alpha_i (H_i x, x) \leqslant 0$ for any $x \in X$); A_0 is a linear completely continuous self-adjoint operator such that $(A_0 x, x) < (x, x)$ for any $x \in X,\ x \neq 0$; A_1 is a linear self-adjoint operator with finite absolute norm. The listed conditions ensure convergence in operator norm, for every $\lambda \neq a_i,\ i = 1, 2, ...$, of the series

$$\sum_{i=1}^{\infty} \frac{1}{\lambda - a_i} H_i.$$

Consider the equation

$$A_\lambda x \equiv x - A_0 x - \lambda \left\{ A_1 x + \lambda \sum_{i=1}^{\infty} \frac{1}{\lambda - a_i} H_i x \right\} = y, \quad x \in X, \ y \in X, \tag{1}$$

and the corresponding homogeneous equation

$$A_\lambda x \equiv x - A_0 x - \lambda \left\{ A_1 x + \lambda \sum_{i=1}^{\infty} \frac{1}{\lambda - a_i} H_i x \right\} = 0, \tag{2}$$

where λ is a complex parameter. A value $\lambda = \lambda_k$ ($\lambda_k \neq a_i, \ i = 1, 2, \ldots$) is called an eigenvalue of equation (2) if the equation $A_{\lambda_k} x = 0$ has a solution $\varphi_k \neq 0$, called an eigenvector. A value $\lambda = a_p$ ($1 \leq p < \infty$) will be called an eigenvalue of equation (2)

in the special sense, if there exist elements $\varphi_k \in X$, $\varphi_q \neq 0$, $\psi \in X$ such that

$$\varphi_k - A_0 \varphi_k - a_0 \left\{ A_1 \varphi_k + a_p \sum_{\substack{i=1 \\ i \neq p}}^{\infty} \frac{1}{a_p - a_i} H_i \varphi_k \right\} + a_p H_p \psi = 0,$$

$$H_p \varphi_k = 0.$$

If $\lambda = \lambda_k$ is an eigenvalue of equation (2), then put

$$\psi_{ik} = -\frac{\lambda_k}{\lambda_k - a_i} \varphi_k, \qquad i = 1, 2, \ldots, \tag{3}$$

whereas if $\lambda = a_p$ is an eigenvalue in the special sense, then put

$$\psi_{ik} = -\frac{a_p}{a_p - a_i} \varphi_k, \quad i \neq p, \quad i = 1, 2, \ldots, \quad \psi_{kp} = \psi. \tag{4}$$

In what follows, both the former and the latter eigenvalues will be called simply eigenvalues.

In the paper [5] we proved a proposition of which the following theorem is a special case.

Theorem 1. *The set of eigenvalues of equation (2) does not condense in the finite part of the λ-plane.*

Equation (2) can have only real eigenvalues. To each eigenvalue there can correspond only a finite number of linearly independent eigenelements. Let $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ be a sequence of eigenvalues of equation (2), each of which is written down as many times as its multiplicity; let $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$ be the corresponding eigenelements, and let $\psi_{1k}, \psi_{2k}, \dots, \psi_{sk}, \dots$ $(k = 1, 2, \dots)$ be understood in accordance with the notations (3) or (4). The sequence of eigenelements can be chosen so that the conditions

$$((E - A_0)\varphi_k, \varphi_n) - \sum_{i=1}^{\infty} a_i (H_i \psi_{ik}, \psi_{in}) = \delta_{kn}; \qquad k, n = 1, 2, \dots, \tag{5}$$

are satisfied, where E is the identity operator.

2. Consider the set G of ordered sequences

$$\alpha \equiv \{y_0,\ y_1, \dots,\ y_n, \dots\}, \qquad y_i \in X\ (i = 0, 1, \dots),$$

satisfying the condition

$$\|\alpha\|_G^2 \equiv ((E - A_0)y_0, y_0) - \sum_{i=1}^{\infty} a_i (H_i y_i, y_i) < +\infty,$$

and define in G the scalar product

$$(\alpha, \alpha^{(1)})_G = ((E - A_0)y_0, y_0^{(1)}) - \sum_{i=1}^{\infty} a_i (H_i y_i, y_i^{(1)}),$$

where

$$\alpha^{(1)} \equiv \{y_0^{(1)},\ y_1^{(1)}, \dots,\ y_n^{(1)}, \dots\}.$$

This product turns G into a Hilbert space, generally speaking not complete, with norm

$$\|\alpha\|_G = (\alpha, \alpha)_G^{1/2}.$$

On the subset $G_1 \in G$ of elements satisfying the condition $\|\alpha\|_G \le 1$, consider the functional

$$\Phi(\alpha) = -(A_1 y_0, y_0) + \sum_{i=1}^{\infty} (H_i y_0, y_i) + \sum_{i=1}^{\infty} (y_i, H_i y_0) + \sum_{i=1}^{\infty} (H_i y_i, y_i)$$

(these series converge absolutely by virtue of our conditions).

Theorem 2. Equation (2) has eigenvalues $|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_n| \le \cdots$, to which there correspond eigen-elements $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$, normalized by conditions (5), possessing the following extremal properties: on the set of elements α satisfying the conditions

$$\|\alpha\|_G = 1, \quad (\alpha, \alpha_k)_G = 0, \quad \alpha_k \equiv \{\varphi_k, \psi_{1k}, \dots, \psi_{sk}, \dots\}, \quad k = 1, \dots, n-1,$$

the absolute value of the functional $\Phi(\alpha)$ attains its greatest value, equal to $1/|\lambda_n|$, which is attained at the element $\alpha_n \equiv \{\varphi_n, \psi_{1n}, \dots, \psi_{sn}, \dots\}$ $(n = 1, 2, \dots)$. This set of eigenvalues (necessarily infinite, if A_1 is an infinite-dimensional operator) exhausts the entire spectrum of equation (2).

3. For the class of linear operators under study, the following spectral expansions hold:

Theorem 3. For any $f \in X$,

$$A_1 f = \sum_{k=1}^{\infty} \frac{(f, (E - A_0)\varphi_k)}{\lambda_k} (E - A_0)\varphi_k,$$

where the series on the right converges in the norm in X.

Theorem 4. For any $f \in X$,

$$H_i f = \sum_{k=1}^{\infty} \frac{a_i (f, H_i \psi_{ik})}{\lambda_k} (E - A_0)\varphi_k, \quad i = 1, 2, \dots,$$

where the series on the right converge in the norm in X.

On the basis of these propositions, it is proved in the usual way that the following theorem is valid.

Theorem 5. If λ is not an eigenvalue of equation (2), then the unique solution of equation (1), for a given $y \in X$, has the form

$$x = (E - A_0)^{-1} y + \lambda \sum_{k=1}^{\infty} \frac{(y, \varphi_k)}{\lambda_k - \lambda} \varphi_k,$$

where the series on the right converges in the norm in X.

4. As above, set $\alpha_n \equiv \{\varphi_n, \psi_{1n}, \dots, \psi_{sn}, \dots\}$, $n = 1, 2 \dots$

$$(\alpha_n \in G, \quad \|\alpha_n\|_G = 1, \quad (\alpha_k, \alpha_n)_G = \delta_{kn}).$$

Theorem 6. If $\alpha = \{y_0, y_1, \dots, y_n, \dots\} \in G$, then

$$A_1 y_0 - \sum_{i=1}^{\infty} H_i y_i = \sum_{n=1}^{\infty} \frac{(\alpha, \alpha_n)_G}{\lambda_n} (E - A_0)\varphi_n,$$

$$H_i y_0 + H_i y_i = \sum_{n=1}^{\infty} \frac{a_i (\alpha, \alpha_n)_G}{\lambda_n} H_i \psi_{in}, \quad i = 1, 2, \dots,$$

where the series converge in the norm in X.

Theorem 7. If, for the element $\alpha = \{y_0, y_1, \dots, y_n, \dots\} \in G$,

$$A_1 y_0 - \sum_{i=1}^{\infty} H_i y_i = 0, \quad H_i y_0 + H_i y_i = 0, \quad i = 1, 2, \dots, \tag{6}$$

then $(\alpha, \alpha_n)_G = 0$, $n = 1, 2, \dots$.

We shall call the system of elements $\{\alpha_n\}$ complete in G if there exists no element $\alpha \in G$ (apart from the zero element) orthogonal to all α_n $(n = 1, 2, \dots)$.

From Theorems 7 and 8 there follows the following criterion of completeness:

Theorem 8. In order that the system of elements $\{\alpha_n\}$ be complete in G, it is necessary and sufficient that equations (6) have only the zero solution

$$y_0 = y_1 = \cdots = y_n = \cdots = 0.$$

The following criterion also holds:

Theorem 9. In order that the system of elements $\{\alpha_n\}$ be complete in G, it is necessary and sufficient that, for every element $\alpha \equiv \{y_0, y_1, \dots, y_n, \dots\} \in G$, the representations

$$y_0 = \sum_{n=1}^{\infty} (\alpha, \alpha_n)_G \varphi_n, \qquad y_i = \sum_{n=1}^{\infty} (\alpha, \alpha_n)_G \psi_{in}, \quad i = 1, 2, \dots, \tag{7}$$

hold, where the series on the right converge in norm in X.

From Theorems 8 and 9 there immediately follows a proposition of important significance for applications to the study of boundary-value problems in the theory of differential equations:

Theorem 10. In order that, for every element $\alpha \equiv \{y_0, y_1, \dots, y_n, \dots\} \in G$, the representations (7) hold, it is necessary and sufficient that equations (6) have only the zero solution.

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Note: Figure translations are in progress. See original paper for figures.